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Bayesian process control for Cronbach's alpha

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In this paper, Bayesian statistical process control limits are derived for Cronbach's coefficient alpha (α) in the case of the balanced one-way random effects model. Cronbach's alpha is one of the most commonly used measures for assessing a set of items' internal consistency or reliability, thereby assessing the assumption that they measure the same latent construct. By using the available data and the Jeffreys independence prior, the posterior distribution of α and the predictive density of a future (unknown) Cronbach's alpha $(\hat{\alpha}_f)$ can be derived. Given a stable Phase I process, the predictive density function $\left(f\left(\hat{\alpha}_f|\hat{\alpha}\right)\right)$ and the conditional predictive density functions $\left(f\left(\hat{\alpha}_f|\alpha\right)\right)$ are used to calculate central values, variances, control limits, run-lengths and the average run-length. The predictive density of a future run-length is the average of a large number of geometrical distributions, each with its own parameter value. Three applications of interest are included in this paper. From the results, it can be seen that the average and median run-lengths are usually larger than the theoretical values. An advantage of the Bayesian procedure, however, is that the control limits, in other words, β, can be adjusted in such a way that the average or median run-length has a specific value.

Keywords: Average run-length, Bayesian analysis, Control limits, Cronbach's alpha, Posterior predictive density, Run-length, Statistical process control.

1. Introduction

Cronbach's coefficient alpha was introduced in Cronbach's (1951) article based on the work of Guttman (1945). It is one of the most commonly used measures for assessing a set of items' internal consistency or reliability, thereby assessing the assumption that they measure the same latent construct. It measures reliability in education, psychology, sociology, medicine, accounting and economics. In economics, for example, researchers might apply it in studies assessing consumer behaviour, while in accounting, they could use it in research related to financial reporting or the auditing of observations. Overall, it helps researchers to ensure that their measurement tools yield reliable data. For more details, see, for example, Hulin et al. (1983), Feldt et al. (1987), Cortina (1993), Kaplan and Saccuzzo (1993), van Zyl et al. (2000), Koning and Franses (2003), Duhachek and Iacobucci (2004) and Izally et al. (2025). According to Cortina (1993), Cronbach's (1951) article was cited approximately sixty times yearly from 1966 to 1990 and in two hundred and seventy-eight

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journals. Recently, it was cited 69 919 times on Google Scholar (Accessed on February 5, 2025). As mentioned by Padilla and Zhang (2011), the reason for the popularity of Cronbach's alpha is that it is computationally simple. Only the sample size and the variance components are needed.

While statistical process control (SPC) techniques, such as control charts, have not been widely applied to Cronbach's alpha (α), they could serve as a valuable method for monitoring whether the reliability of a test remains stable across different samples or time points. If variation in alpha is observed across different samples or time points, SPC could help determine whether this variability is attributable to natural variation or special causes.

Since Bayesian statistical process control (SPC) techniques applied to Cronbach's alpha are even less common than classical SPC methods, and given that Cronbach's coefficient alpha is a key statistic in the field of reliability research, this study implements Bayesian SPC procedures for monitoring alpha.

Statistical process control (SPC) refers to statistical procedures and problem-solving methods used to control and monitor the quality of the output of a production process. SPC aims to detect and eliminate uncontrolled variation in the process. For more information, see, for example, Balakrishnan et al. (2006), Montgomery (2005) and Human (2009). Statistical process control usually involves two phases. During Phase I, the system establishes control limits, and in Phase II, the system monitors the process to detect any breaches of these limits. van Zyl and van der Merwe (2019) indicate that the conventional approach to statistical process control is frequentist, where a Phase I study estimates unknown parameters using maximum likelihood estimates. The system uses these estimates to model distributions in Phase II. The Bayesian framework argues that deriving the true underlying parameter values in such a way is unsatisfactory. Instead, it derives the posterior distributions of the unknown parameters by using a prior distribution. The posterior distributions illustrate the uncertainties in the parameter values and incorporate them through predictive distributions.

Several Bayesian process monitoring schemes use predictive distributions. See, for example, Menzefricke (2002, 2007, 2010a, 2010b) and van Zyl and van der Merwe (2019). Also, in Tsiamyrtzis and Hawkins (2006), the following is mentioned: "A particularly interesting feature in the Bayesian paradigm is forecasting. Namely, one can use the available data to derive the predictive distribution of the next (unseen) observation". In this paper, it will be the next (unseen) Cronbach's alpha ($\hat{\alpha}_f$). Given a stable Phase I process, the predictive distribution $\left(f\left(\hat{\alpha}_f|\text{data}\right)\right)$ will be used to calculate central values, variances, prediction intervals, control limits, the run-length and the average run-length of a future Cronbach's alpha. Bayarri and García-Donato (2005) highlighted the following reasons for recommending a Bayesian analysis for control charts: (i) Bayesian methods allow naturally for prediction and control charts rely on future observations. (ii) Objective Bayesian procedures are possible without introducing information other than the model. (iii) The numerical difficulties of a Bayesian procedure are easily handled via Monte Carlo simulation. The unconditional predictive distribution of a future Cronbach's coefficient, $\hat{\alpha}_f$, will be obtained using Monte Carlo simulation and the Rao-Blackwell procedure or numerical integration. The advantage of a Bayesian approach to process monitoring arises from the sequential nature of Bayes' theorem. As pointed out by Tsiamyrtzis and Hawkins (2006), Shiau and Feltz (2006), Alt (2006), Tagaras and Nenes (2006) and Graves (2006), a Bayesian approach allows a more flexible framework, in particular concerning the usual assumption made in SPC charts about known parameters. The above-mentioned authors considered both univariate and multivariate process monitoring techniques. They discuss the applications and development of full Bayesian approaches and empirical Bayesian methods.

The researchers Peterson (2006) and Moreno (2006) focussed on Bayesian methods for process optimisation. According to them, the predictive approach to response surface optimisation represented a major advance in response surface method techniques as it incorporates the uncertainty of the parameter estimates in the optimisation process. They also mentioned that this has no frequentist counterpart.

Section 2 derives the model, the choice of prior, and the posterior distribution of α. Section 3 provides an example and shows how to obtain the predictive density function of a future (unseen) Cronbach's alpha using numerical integration or the Rao-Blackwell simulation procedure. In Section 4, the run-length and average run-length are discussed. The predictive density function of a future run-length is the average of a large number of geometric distributions, each with its own parameter value. Section 5 calculates the posterior and predictive densities for a larger example, and Section 6 considers the "In Control" and "Out of Control" situations. Section 7 analyses an example of measured values of "Bore diameter". Conclusions are given in Section 8.

2. The balanced random effects model

2.1 The model

The model that will be used is the balanced one-way random effects (variance components) model

$$Y_{ij} = \theta + r_i + \epsilon_{ij} \quad \text{for } i = 1, ..., I, \text{ and } j = 1, ...J, \tag{1}$$

where Y_{ij} is the value corresponding to the jth observation made at the ith group (sample). θ is a constant referred to as the overall mean, and it is unknown. The r_i and ϵ_{ij} are independent normal variables with zero means and variances σ_r^2 and σ_ϵ^2, respectively. Let $\mathbf{Y_i} = [Y_{i1}, Y_{i2}, ...Y_{iJ}]'$ be the observations associated with group i. It can be shown that $Var\left(\mathbf{Y}|\theta, \sigma_\epsilon^2, \sigma_r^2\right) = \sigma_\epsilon^2 \tilde{\mathbf{I}} + \mathbf{11}'\sigma_r^2 = \mathbf{\Sigma}$ where $\tilde{\mathbf{I}}$ is the $J \times J$ identity matrix and $\mathbf{1} = [1\ 1 ... 1]'$ is a $J \times 1$ column vector of ones. The model in Equation 1 is called the balanced one-way random effects model because the number of observations, J, in each sample are the same. The covariance matrix is given by

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_\epsilon^2 + \sigma_r^2 & \sigma_r^2 & \sigma_r^2 & \dots & \sigma_r^2 \\ \sigma_r^2 & \sigma_\epsilon^2 + \sigma_r^2 & \sigma_r^2 & \dots & \sigma_r^2 \\ \sigma_r^2 & \dots & \sigma_\epsilon^2 + \sigma_r^2 & \cdots & \sigma_r^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_r^2 & \sigma_r^2 & \sigma_r^2 & \cdots & \sigma_\epsilon^2 + \sigma_r^2 \end{bmatrix}_{J \times J}.$$

The covariance matrix is called compound symmetric since all the variances along the diagonal are the same, namely $\sigma_\epsilon^2 + \sigma_r^2$ and all the covariances σ_r^2 are equal. A general definition for α is

$$\alpha = \frac{J}{(J-1)} \left\{1 - \frac{tr\,(\mathbf{\Sigma})}{\mathbf{1}'\mathbf{\Sigma}\mathbf{1}}\right\},$$

where $tr\,(\mathbf{\Sigma})$ is the sum of the covariance matrix's diagonal elements (variances). For the random effects model

$$\begin{aligned}\alpha =&\frac{J}{(J-1)}\left\{1-\frac{J\left(\sigma_\epsilon^2+\sigma_r^2\right)}{J\left(\sigma_\epsilon^2+\sigma_r^2\right)+J\left(J-1\right)\sigma_r^2}\right\}\\ =&\frac{J\sigma_r^2}{\sigma_\epsilon^2+J\sigma_r^2}=1-\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2+J\sigma_r^2}.\end{aligned} \tag{2}$$

2.2 The prior and posterior distribution of α

In Box and Tiao (1973), it is shown that the likelihood function for the model defined in Equation 1 is

$$\ell(\theta,\sigma_\epsilon^2,\sigma_r^2|data)\propto\left(\sigma_\epsilon^2\right)^{-v_1/2}\left(\sigma_\epsilon^2+J\sigma_r^2\right)^{-(v_2+1)/2} \times\exp\left\{-\frac{1}{2}\left[\frac{IJ\left(\bar{Y}_{..}-\theta\right)^2}{\left(\sigma_\epsilon^2+J\sigma_r^2\right)}+\frac{v_2m_2}{\left(\sigma_\epsilon^2+J\sigma_r^2\right)}+\frac{v_1m_1}{\sigma_\epsilon^2}\right]\right\},$$

where $v_1=I\left(J-1\right), v_2=I-1, \bar{Y}_{i.}=\frac{1}{J}\sum_{j=1}^J Y_{ij}, \bar{Y}_{..}=\frac{1}{IJ}\sum_{i=1}^I\sum_{j=1}^J Y_{ij}, v_1m_1=\sum_{i=1}^I\sum_{j=1}^J(Y_{ij}-\bar{Y}_{i.})^2$ and $v_2m_2=J\sum_{i=1}^I(\bar{Y}_{i.}-\bar{Y}_{..})^2$. The quantity v_1m_1 is the within group sums of squares and v_2m_2 is the between groups sums of squares. Since $E\left(m_1\right)=\sigma_\epsilon^2$ and $E\left(m_2\right)=\sigma_\epsilon^2+J\sigma_r^2$ an estimate of α is

$$\hat{\alpha}=1-\frac{m_1}{m_2}.$$

The prior that will be used in the analysis, is the Jeffreys independence prior

$$p\left(\sigma_\epsilon^2,\sigma_r^2\right)\propto\left(\sigma_\epsilon^2\right)^{-1}\left(\sigma_\epsilon^2+J\sigma_r^2\right)^{-1}.$$

It can be shown that it is also a reference and probability matching prior. By multiplying the likelihood with the prior and integrating with respect to θ the posterior density function of the variance components is given by

$$p(\sigma_\epsilon^2,\sigma_r^2|data)\propto\left(\sigma_\epsilon^2\right)^{-(v_1+2)/2}\left(\sigma_\epsilon^2+J\sigma_r^2\right)^{-(v_2+2)/2}\exp\left\{-\frac{1}{2}\left[\frac{v_1m_1}{\sigma_\epsilon^2}+\frac{v_2m_2}{\sigma_\epsilon^2+J\sigma_r^2}\right]\right\}, \tag{3}$$

for $\sigma_\epsilon^2>0,\sigma_r^2>0$. From Equation 3 it is clear that $p(\sigma_\epsilon^2,\sigma_\epsilon^2+J\sigma_r^2|data)=p(\sigma_\epsilon^2|data)p(\sigma_\epsilon^2+J\sigma_r^2|data)$. Since the posterior distributions of σ_ϵ^2 and $\sigma_\epsilon^2+J\sigma_r^2$ are independent inverse-gamma distributions, it follows that

$$\sigma_\epsilon^2\sim\frac{v_1m_1}{\chi_{v_1}^2}\quad\text{and}\quad\sigma_\epsilon^2+J\sigma_r^2\sim\frac{v_2m_2}{\chi_{v_2}^2}.$$

The posterior density function of $\alpha=1-\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2+J\sigma_r^2}$ can therefore easily be obtained and is given in Theorem 1.

Theorem 1. *The posterior density function of α is given by*

$$p\left(\alpha|\hat{\alpha}\right)=K_1\left(\frac{v_2}{v_1}\right)^{\frac{1}{2}v_2}\left(\frac{1}{1-\hat{\alpha}}\right)^{\frac{1}{2}v_2}\left(1-\alpha\right)^{\frac{1}{2}v_2-1}\times\left[1+\frac{v_2}{v_1}\left(\frac{1-\alpha}{1-\hat{\alpha}}\right)\right]^{-\frac{1}{2}(v_1+v_2)},$$

where

$$K_1 = \frac{\Gamma\left(\frac{v_1+v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)}$$

and $\hat{\alpha} = 1 - \frac{m_1}{m_2}$. *Since*

$$\alpha|\hat{\alpha} \sim 1 - (1-\hat{\alpha})\, F_{v_2,v_1}, \tag{4}$$

$$E(\alpha|\hat{\alpha}) = 1 - (1-\hat{\alpha})\left(\frac{v_1}{v_1-2},\right)$$

and $Var(\alpha|\hat{\alpha}) = (1-\hat{\alpha})^2 \frac{2v_1^2(v_2+v_1-2)}{v_2(v_1-2)^2(v_1-4)}$.

Proof. The proof is given in Appendix A. ■

3. Simulation procedure of the predictive density function

3.1 The predictive density function of a future Cronbach's α

Consider a future (unseen) experiment

$$\tilde{Y}_{ij} = \theta + r_i + \epsilon_{ij} \quad \text{for } i = 1, \ldots \tilde{I} \text{ and } j = 1, \ldots J$$

where $\tilde{Y}_{ij}$ is the value of the j^{th} observation in the i^{th} group (sample). The number of samples in the future experiment $\left(\tilde{I}\right)$ can differ from those in the original experiment (data set) (I). However, the number of observations per sample is the same in both experiments, namely J. As before, the r_i and ϵ_{ij} are independent normal variables with zero means and variances σ_r^2 and σ_ϵ^2, respectively. Define

$$\tilde{v}_1\tilde{m}_1 = \sum_{i=1}^{\tilde{I}}\sum_{j=1}^{J}(\tilde{Y}_{ij} - \bar{\tilde{Y}}_{i.})^2$$

and

$$\tilde{v}_2\tilde{m}_2 = J\sum_{i=1}^{\tilde{I}}(\bar{\tilde{Y}}_{i.} - \bar{\tilde{Y}}_{..})^2$$

where $\tilde{v}_1 = \tilde{I}(J-1)$, $\tilde{v}_2 = \tilde{I} - 1$, $\bar{\tilde{Y}}_{i.} = \frac{1}{J}\sum_{j=1}^{J}\tilde{Y}_{ij}$, $\bar{\tilde{Y}}_{..} = \frac{1}{\tilde{I}J}\sum_{i=1}^{\tilde{I}}\sum_{j=1}^{J}\tilde{Y}_{ij}$. The quantity $\tilde{v}_1\tilde{m}_1$ is the within group sums of squares and $\tilde{v}_2\tilde{m}_2$ is the between groups sums of squares of the new (unseen) experiment (data set). It is well known from classical (traditional) statistics that $E(\widetilde{m}_1) = \sigma_\epsilon^2$ and $E(\widetilde{m}_2) = \sigma_\epsilon^2 + J\sigma_r^2$. A future (unseen) Cronbach's alpha is therefore defined as

$$\hat{\alpha}_f = 1 - \frac{\widetilde{m}_1}{\widetilde{m}_2}.$$

For given σ_ϵ^2 and $\sigma_\epsilon^2 + J\sigma_r^2$, it follows that $\frac{\tilde{v}_1\widetilde{m}_1}{\sigma_\epsilon^2} \sim \chi^2_{\tilde{v}_1}$ and $\frac{\tilde{v}_2\widetilde{m}_2}{\sigma_\epsilon^2+J\sigma_r^2} \sim \chi^2_{\tilde{v}_2}$. Therefore $\widetilde{m}_1 \sim \sigma_\epsilon^2\frac{\chi^2_{\tilde{v}_1}}{\tilde{v}_1}$, $\widetilde{m}_2 \sim \left(\sigma_\epsilon^2 + J\sigma_r^2\right)\frac{\chi^2_{\tilde{v}_2}}{\tilde{v}_2}$ and

$$\frac{\widetilde{m}_1}{\widetilde{m}_2} \sim \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + J\sigma_r^2}F_{\tilde{v}_1,\tilde{v}_2}.$$

Also $\hat{\alpha}_f = 1 - \frac{\widetilde{m}_1}{\widetilde{m}_2} \sim 1 - \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + J\sigma_r^2} F_{\tilde{v}_1, \tilde{v}_2}$ and therefore

$$\hat{\alpha}_f | \alpha \sim 1 - (1 - \alpha) F_{\tilde{v}_1, \tilde{v}_2} \tag{5}$$

where $F_{\tilde{v}_1, \tilde{v}_2}$ is an F distribution with $\tilde{v}_1$ and $\tilde{v}_2$ degrees of freedom. The following theorem can now be proved.

Theorem 2. *The predictive density of a future Cronbach's alpha* $(\hat{\alpha}_f)$, *for given* α *is*

$$f\left(\hat{\alpha}_f | \alpha\right) = K_2 \left(\frac{\tilde{v}_1}{\tilde{v}_2}\right)^{\frac{\tilde{v}_1}{2}} \left(\frac{1}{1-\alpha}\right)^{\frac{\tilde{v}_1}{2}} \left(1 - \hat{\alpha}_f\right)^{\frac{1}{2}\tilde{v}_1 - 1} \left[1 + \frac{\tilde{v}_1}{\tilde{v}_2}\left(\frac{1 - \hat{\alpha}_f}{1 - \alpha}\right)\right]^{-\frac{1}{2}(\tilde{v}_1 + \tilde{v}_2)} \tag{6}$$

where $K_2 = \frac{\Gamma\left(\frac{\tilde{v}_1 + \tilde{v}_2}{2}\right)}{\Gamma\left(\frac{\tilde{v}_1}{2}\right)\Gamma\left(\frac{\tilde{v}_2}{2}\right)}$. *Also,*

$$E\left(\hat{\alpha}_f | \alpha\right) = 1 - (1 - \alpha)\left(\frac{\tilde{v}_2}{\tilde{v}_2 - 2}\right)$$

and $Var\left(\hat{\alpha}_f | \alpha\right) = (1 - \alpha)^2 \frac{2(\tilde{v}_2)^2 (\tilde{v}_2 + \tilde{v}_1 - 2)}{\tilde{v}_1 (\tilde{v}_2 - 2)^2 (\tilde{v}_2 - 4)}$.

Proof. The proof is given in Appendix B. ■

3.2 The unconditional predictive density function of a future Cronbach's alpha $(\hat{\alpha}_f)$

The predictive density of a future Cronbach's alpha is

$$f\left(\hat{\alpha}_f | \hat{\alpha}\right) = \int f\left(\hat{\alpha}_f | \alpha\right) p\left(\alpha | \hat{\alpha}\right) d\alpha, \tag{7}$$

where $p(\alpha|\hat{\alpha})$ is the posterior density function of α derived in Theorem 1 and $f\left(\hat{\alpha}_f|\alpha\right)$ is the conditional predictive density function derived in Theorem 2. The integral in Equation 7 is difficult to solve analytically, but can be obtained by either numerical integration or the following simulation procedure:

1. Calculate $\hat{\alpha} = 1 - \frac{m_1}{m_2}$.
2. Simulate α from its posterior distribution and substitute it in $f\left(\hat{\alpha}_f|\alpha\right)$ and draw the density function. In the proof of Theorem 1 it is shown that

   $$\alpha | \hat{\alpha} \sim 1 - (1 - \hat{\alpha}) F_{v_2, v_1}. \tag{8}$$

   It is, therefore, easy to simulate α from its posterior distribution.
3. Iterate step two 100 000 times and determine the average of the 100 000 conditional predictive density functions to obtain $f\left(\hat{\alpha}_f|\hat{\alpha}\right)$, the unconditional predictive density function. This method is called the Rao-Blackwell procedure. $\hat{\alpha}$ represents the data.
4. Determine the mean, median, mode and variance of $f\left(\hat{\alpha}_f|\hat{\alpha}\right)$ as well as the 90^{th} and 95^{th} prediction intervals. For the Dyestuff data, it will first be assumed that $\tilde{v}_1 = v_1 = 24$ and $\tilde{v}_2 = v_2 = 5$. In other words, future experiments will have the same number of samples and observations per sample as the original data set.

Table 1. Dyestuff data.

Batch					
1	**2**	**3**	**4**	**5**	**6**
1545	1540	1595	1445	1595	1520
1440	1555	1550	1440	1630	1455
1440	1490	1605	1595	1515	1450
1520	1560	1510	1465	1635	1480
1580	1495	1560	1545	1625	1445

For the predictive density function $f\left(\hat{\alpha}_f|\hat{\alpha}\right)$, the exact mean and variance can be derived analytically. The following theorem can now be stated.

Theorem 3. *The mean and variance of $f\left(\hat{\alpha}_f|\hat{\alpha}\right)$, the unconditional predictive density function of $\hat{\alpha}_f$, are given by*

$$E\left(\hat{\alpha}_f|\hat{\alpha}\right) = 1-(1-\hat{\alpha})\left(\frac{v_1}{v_1-2}\right)\left(\frac{\tilde{v}_2}{\tilde{v}_2-2}\right) \tag{9}$$

and

$$\begin{aligned} Var\left(\hat{\alpha}_f|\hat{\alpha}\right) &= (1-\hat{\alpha})^2\left\{Var\left(F_{v_2,v_1}\right)+\left(\frac{v_1}{v_1-2}\right)^2\right\}Var\left(F_{\tilde{v}_1,\tilde{v}_2}\right) \\ &\quad +(1-\hat{\alpha})^2\left(\frac{\tilde{v}_2}{\tilde{v}_2-2}\right)^2 Var\left(F_{v_2,v_1}\right), \end{aligned} \tag{10}$$

where $Var\left(F_{a,b}\right) = \frac{2b^2(a+b-2)}{a(b-2)^2(b-4)}$.

Proof. The proof is given in Appendix C. ∎

3.3 Example: The Dyestuff data

Consider the following example from Box and Tiao (1973), which concerns dyestuff data. The experiment was aimed at learning to what extent batch to batch variation in a certain raw material was responsible for variation in the final product yield. Five samples from each of six randomly chosen batches of raw material were taken, and a single laboratory determination of product yield was made for each of the resulting 30 samples. The data is from Davies and Goldsmith (1972), where they reported the data from an experiment designed to investigate the batch to batch variation in the quality of an intermediate product (H-acid) on the yield of a dyestuff (Naphthlence Black 1213) made from it. Six samples of the H-acid representing different batches of works manufactured were selected and five preparations of the dyestuff were made in the laboratory for each sample. The equivalent yields of each preparation as grams of standard colour were determined by dye-trial, and the data are given in Table 1 (Sahai and Ojeda, 2004). This is a repetitive process. In this example, $I = 6$ refers to the number of batches, and $J = 5$ denotes the number of observations contained within each batch.

Table 2. Summary statistics and credibility intervals of α.

Mean (α)	Median (α)	Mode (α)	Var (α)
0.7666	0.8051	0.879	0.0266
90% HPD interval		95% HPD interval	
(0.5187; 0.9705)		(0.3990; 0.9746)	

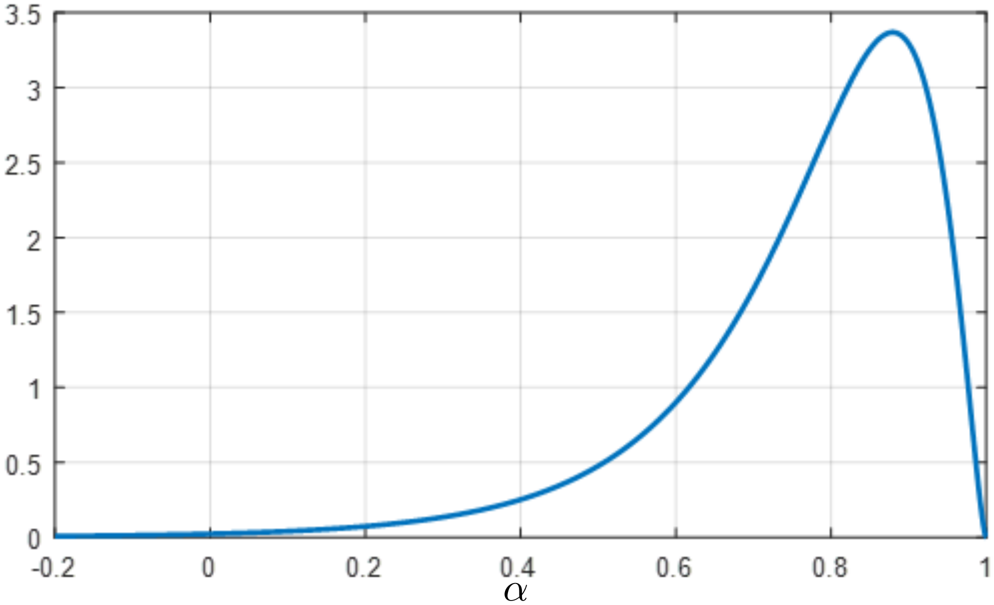

Figure 1. Posterior density function $p\left(\alpha|\hat{\alpha}\right)$ in the case of the Dyestuff data.

From the data, it follows that $v_1 m_1 = \sum_{i=1}^{I}\sum_{j=1}^{J}(y_{ij} - \bar{y}_{i.})^2 = 58\,830$ and $v_2 m_2 = J\sum_{i=1}^{I}(\bar{y}_{i.} - \bar{y}_{..})^2 = 56\,358$ and $\hat{\alpha} = 1 - \frac{m_1}{m_2} = 0.7825268$. The posterior distribution of α which is derived in Theorem 1 is illustrated for the Dyestuff data in Figure 1.

The values for the Mean $(\alpha) = 0.7666$ and Var $(\alpha) = 0.0266$ correspond well with the theoretical values

$$E\left(\alpha|\hat{\alpha}\right) = 1 - \left(1 - \hat{\alpha}\right)\left(\frac{v_1}{v_1 - 2}\right) = 0.7627564$$

and

$$Var\left(\alpha|\hat{\alpha}\right) = \left(1 - \hat{\alpha}\right)^2 \frac{2v_1^2\left(v_2 + v_1 - 2\right)}{v_2\left(v_1 - 2\right)^2\left(v_1 - 4\right)} = 0.0303936.$$

In Figure 2, the predictive density $f\left(\hat{\alpha}_f|\hat{\alpha}\right)$ of a future Cronbach's alpha is illustrated. For the Dyestuff data, $E\left(\hat{\alpha}_f|\hat{\alpha}\right) = 0.6045941$ and $Var\left(\hat{\alpha}_f|\hat{\alpha}\right) = 0.6261647$. The numerical values of the mean and variance given in Table 3 are for all practical purposes the same as the exact values.

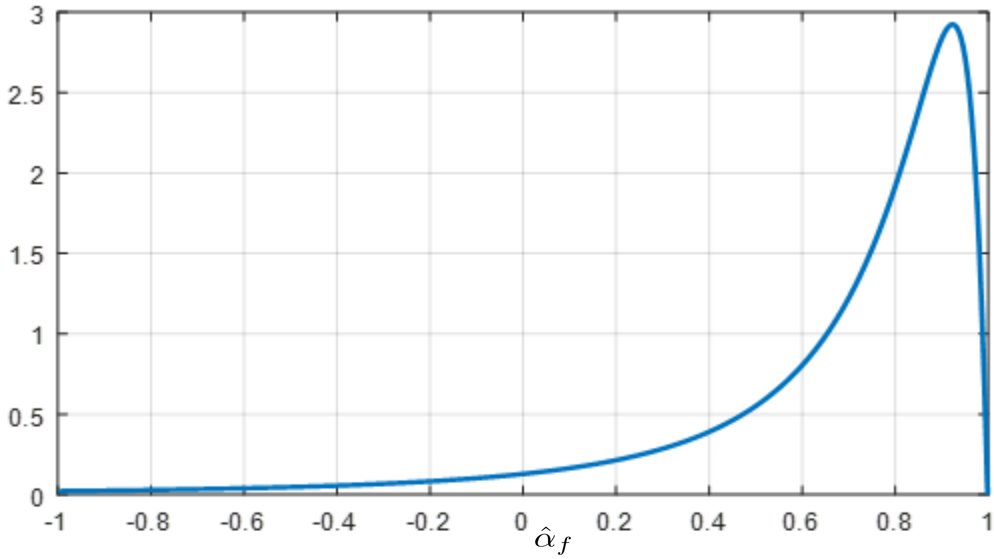

Figure 2. Predictive density function $f\left(\hat{\alpha}_f|\hat{\alpha}\right)$ for the Dyestuff data.

Table 3. Summary statistics and credibility intervals of $\hat{\alpha}_f$ for the Dyestuff data, $\beta = 0.1$.

Mean $\left(\hat{\alpha}_f\right)$	Median $\left(\hat{\alpha}_f\right)$	Mode $\left(\hat{\alpha}_f\right)$	Var $\left(\hat{\alpha}_f\right)$
0.6038	0.780	0.922	0.6023
90% Equal-tailed interval		95% HPD interval	
$(-0.275; 0.9638)$		$(-0.332; 0.989)$	

4. Run-length distribution illustration

Assuming that the process remains stable, the predictive distribution of $\hat{\alpha}_f$ can be used to derive the run-length and average run-length distribution. Montgomery (1996) defines the average run length as the average number of points that must be plotted before a point indicates an out-of-control condition. The run-length (r) is the number of future $\hat{\alpha}_f$ values until the process goes out of control (until the control charts signal for the first time). Note that r does not include the $\hat{\alpha}_f$ value when the control chart signals.

The resulting region of size β for the determination of the run-length is given by

$$\beta = \int_{R(\beta)} f\left(\hat{\alpha}_f\,|\hat{\alpha}\right) d\hat{\alpha}_f, \tag{11}$$

where $f\left(\hat{\alpha}_f\,|\hat{\alpha}\right)$ is defined in Equation 7.

In the case of the 90% prediction interval with $\beta = 0.10$, $R(\beta)$ presents those values of $\hat{\alpha}_f$ that are smaller than -0.275 and larger than 0.9628. The process goes out of control if $\hat{\alpha}_f$ is less than -0.275 or larger than 0.9628. Given α and a stable Phase I process, the distribution of the run-length is geometrical with parameter

$$\Psi(\alpha) = \int_{R(\beta)} f\left(\hat{\alpha}_f \,|\alpha\right) d\hat{\alpha}_f, \tag{12}$$

where $f\left(\hat{\alpha}_f \,|\alpha\right)$ is the distribution of a future $\hat{\alpha}_f$ value given that α is known; see Theorem 2, Equation 6. The values of α are however, unknown, and its uncertainty is described by the posterior distribution given in Theorem 1. By simulating α from the posterior distribution and substituting it in Equation 12, $\Psi(\alpha)$ can be calculated. This procedure must be done for each future experiment. Therefore by simulating a large number of α values, a large number of $\Psi(\alpha)$ values can be obtained. Also a large number of geometric distributions, i.e., a large number of run-length distributions, each with a different set of parameter values $\left\{\Psi\left(\alpha^{(1)}\right), \Psi\left(\alpha^{(2)}\right), ..., \Psi\left(\alpha^{(m)}\right)\right\}$ will be obtained. Since the run-length r for given α is geometrical distributed with mean

$$E(r|\alpha) = \frac{1-\Psi(\alpha)}{\Psi(\alpha)}$$

and variance

$$Var(r|\alpha) = \frac{1-\Psi(\alpha)}{\Psi^2(\alpha)},$$

the unconditional mean

$$E(r\,|\hat{\alpha}) = E\{E(r|\alpha)\}$$

and the unconditional variance

$$Var(r\,|\hat{\alpha}) = E\{Var(r|\alpha)\} + Var\{E(r|\alpha)\}$$

can be calculated. The expectation and variance are taken with respect to the posterior distribution $p(\alpha\,|\hat{\alpha})$.

In Figure 3, the distribution of the geometric parameter $\Psi(\alpha)$ is given and the summary statistics and credibility intervals are given in Table 4.

As it should be, the mean of 0.0989 is in the line with $\beta = 0.1$. The long tail of the above distribution indicates the uncertainty in the parameter $\Psi(\alpha)$. In Figure 4, the distribution of the expected run-length $E(r|\alpha)$ is illustrated. Summary statistics and credibility intervals are given in Table 5. In Figure 5, the distribution of $Var(r|\alpha)$ is given and in Figure 6, the predictive density function of a future run-length, which is the average of a large number of geometric distributions

Table 4. Summary statistics and credibility intervals of $\Psi(\alpha)$.

Mean ($\Psi(\alpha)$)	Median ($\Psi(\alpha)$)	Mode ($\Psi(\alpha)$)	Var ($\Psi(\alpha)$)
0.0989	0.0545	0.0228	0.0149
90% Equal-tailed interval		95% HPD interval	
(0.0240; 0.4816)		(0.0239; 0.3185)	

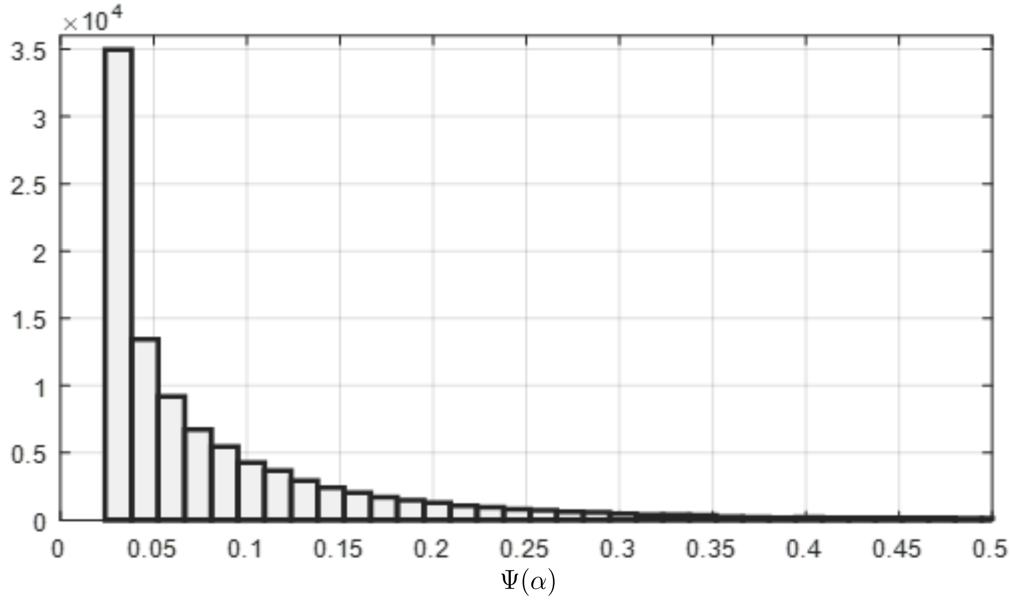

Figure 3. Histogram of $\Psi\ (\alpha)$.

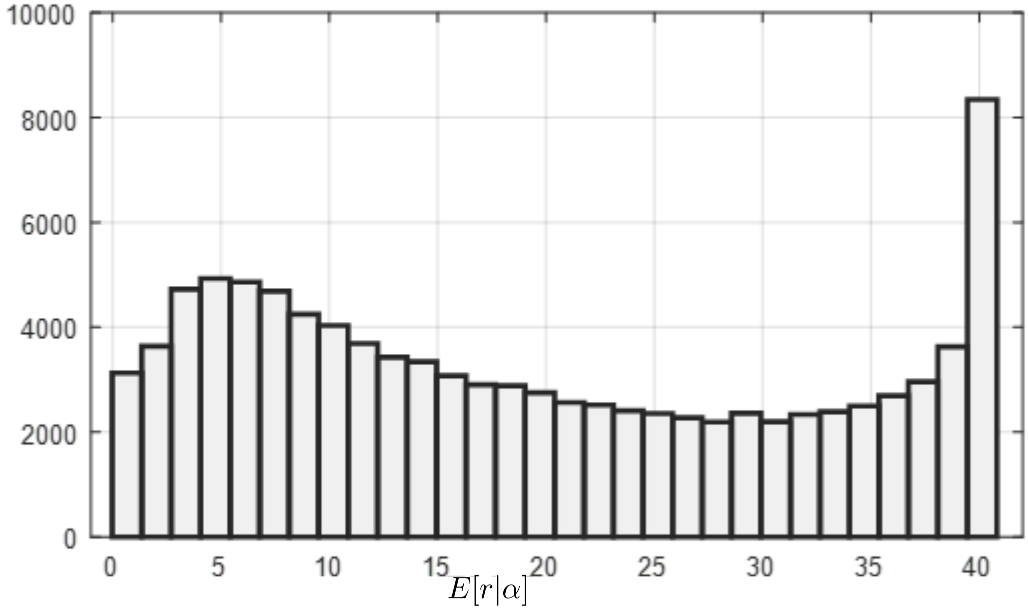

Figure 4. Distribution of the expected run-length $E\ [r\ |\alpha\ ]$ in the case of the 90% prediction interval.

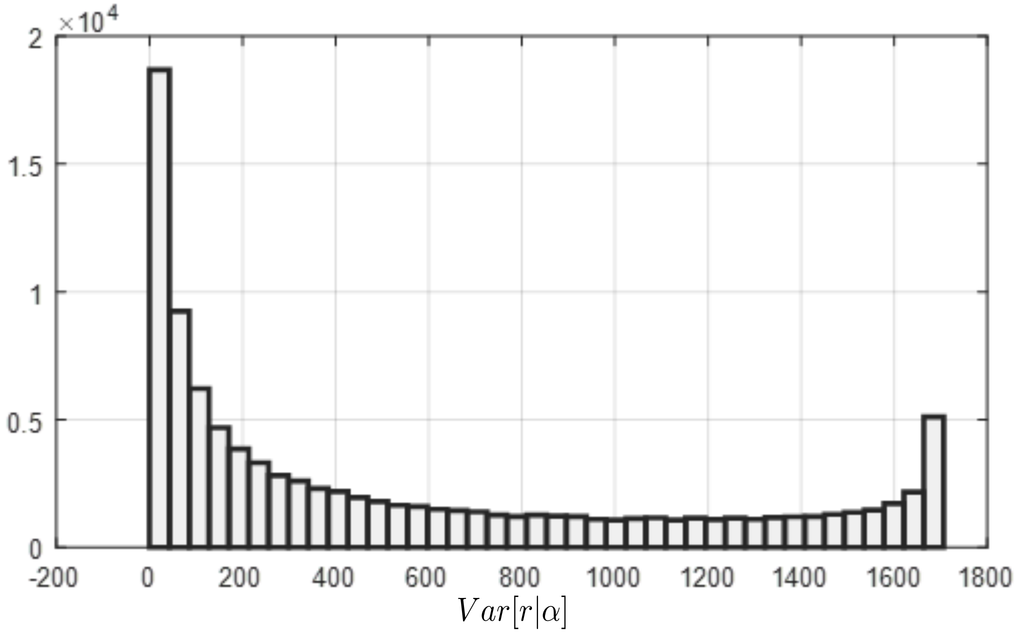

Figure 5. Histogram of $Var\ (r\ |\alpha\ )$.

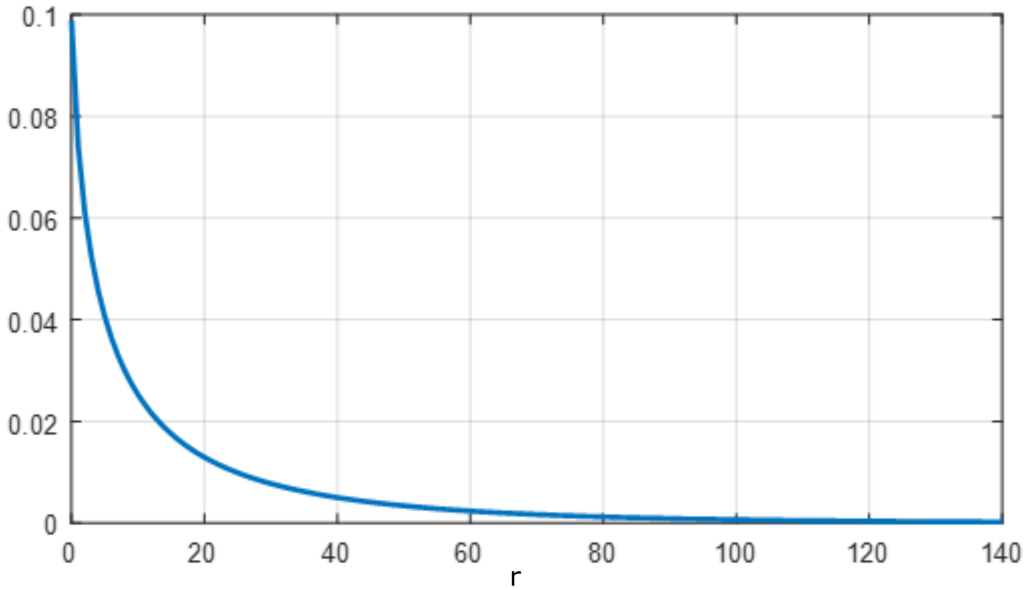

Figure 6. The predictive density function $f\ (r\ |\hat{\alpha}\ )$ in the case of the 90% prediction interval ($\beta\ = 0.1$).

Table 5. Summary statistics and credibility intervals of $E\,[r|\alpha]$.

Mean $(E\,[r|\alpha])$	Median $(E\,[r|\alpha])$	Mode $(E\,[r|\alpha])$	Var $(E\,[r|\alpha])$
19.4147	17.3609	40	164.2296
95% HPD interval (2.1222; 40.8112)			

Table 6. Summary statistics and credibility intervals of r.

Mean $(r\,|\hat{\alpha})$	Median $(r\,|\hat{\alpha})$	Var $(r\,|\hat{\alpha})$
19.4113	9.01	723.41
90% HPD interval (0; 50.33)		95% HPD interval (0; 72.27)

each with its own parameter value, $\Psi\,(\alpha)$, is given. Summary statistics and credibility intervals are given in Table 6.

A comparison of Figures 4 and 6 shows that $Mean\,(E\,[r\,|\alpha\,]) = 19.4147$ and $Mean\,(r\,|\hat{\alpha}\,) = 19.4113$. The means are for all practical purposes the same. Theoretically, this should have been the case. Also $Var\,(r|\hat{\alpha}) = E\,\{Var\,(r|\alpha)\} + Var\,\{E\,(r|\alpha)\} = 560.5730 + 164.22 = 724.8026$. From Table 6, it is clear that $Var\,(r|\hat{\alpha}) = 723.41$. The two methods of calculating $Var\,(r|\hat{\alpha})$ give for all practical purposes the same answer. A mean run-length of 19.41 is larger than $\frac{1-0.1}{0.1} = 9$, which would have been expected if $\beta = 0.1$. The reason for this larger average run-length is the small parameter values of some of the geometric distributions. The median run-length of 9.01 however corresponds well with the theoretical value of 9. A mean run-length of 19.4 is an indication that the charting statistic $(\hat{\alpha}_f)$ will signal on average every 19^{th} or 20^{th} experiment even if the Phase I process is stable (in control). It can however take much longer. According to the 95% HPD interval, it can take as long as 72 experiments. It is however not impossible that $\hat{\alpha}_f$ will signal as early as the first experiment.

5. Example: A larger experiment

In the example of the Dyestuff data, it was mentioned that it is a repetition process. So let us assume that after a certain time period the number of samples obtained were 120 and the number of observations per sample was five, which means that $I = 120$, $J = 5$, $v_1 = I\,(J-1) = 480$, $v_2 = I - 1 = 119$ and $\hat{\alpha} = 0.7825$. In Figure 7 the posterior distribution of α is illustrated. Using the formulas derived in Theorem 1, the exact mean and variance can be calculated $E\,(\alpha|\hat{\alpha}) = 0.7815899$

Table 7. Summary statistics and credibility intervals of α.

Mean (α)	Median (α)	Mode (α)	Var (α)
0.7816	0.7834	0.786	0.0010
90% Equal-tailed interval (0.7264; 0.8306)		95% Equal-tailed interval (0.7143; 0.8386)	

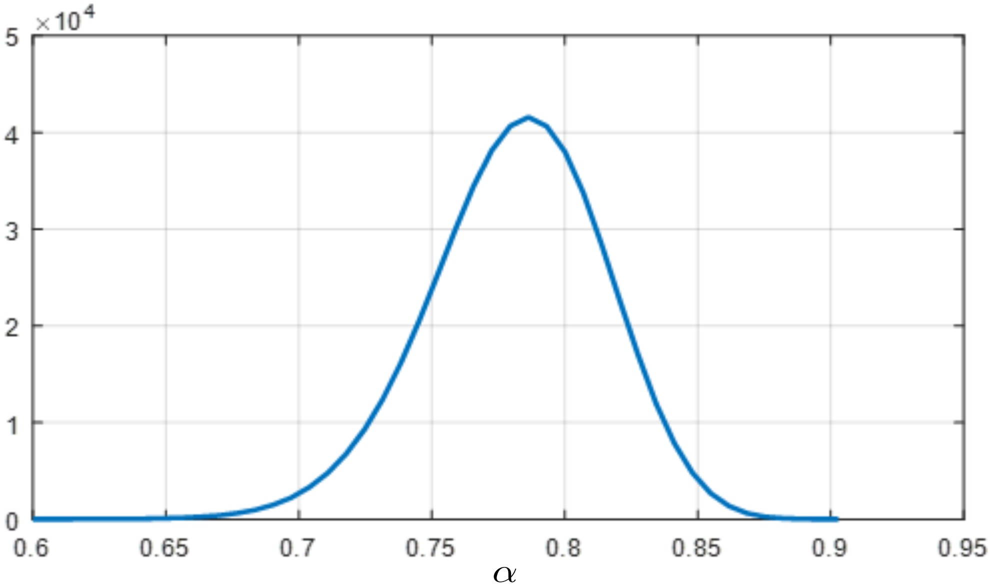

Figure 7. Posterior density function $p(\alpha|\hat{\alpha})$ with $v_1 = 480$, $v_2 = 119$ and $\hat{\alpha} = 0.7825$.

Table 8. Summary statistics and credibility intervals of $\hat{\alpha}_f$ for the Larger Data, $\beta = 0.1$.

Mean $(\hat{\alpha}_f)$	Median $(\hat{\alpha}_f)$	Mode $(\hat{\alpha}_f)$	Var $(\hat{\alpha}_f)$
0.7765	0.7822	0.79	0.0025

90% Equal-tailed Interval	95% HPD Interval
(0.6854; 0.8486)	(0.6763; 0.8677)

and $Var(\alpha|\hat{\alpha}) = 0.0010055$. The theoretical values are for all practical purposes the same as the numerical values, $Mean(\alpha) = 0.7816$ and $Var(\alpha) = 0.0010$. The predictive density function of $\hat{\alpha}_f$ for a future (unseen) experiment consisting of 90 samples and five observations per sample ($\tilde{I} = 90$, $J = 5$, $\tilde{v}_1 = \tilde{I}(J-1) = 360$, $\tilde{v}_2 = \tilde{I} - 1 = 89$ and $\hat{\alpha} = 0.7825$) is displayed in Figure 8.

The formulas for the exact mean and variance are derived in Theorem 3, which is given by $E(\hat{\alpha}_f|\hat{\alpha}) = 0.776569$ and $Var(\hat{\alpha}_f|\hat{\alpha}) = 0.0025414$. The theoretical values are similar to those given in Table 8, which are equal to $Mean(\hat{\alpha}_f) = 0.7765$ and $Var(\hat{\alpha}_f) = 0.0025$. The posterior and predictive density functions illustrated in Figures 7 and 8 are much more symmetrical than those shown in Figures 1 and 2. The reason for this is the larger sample sizes. Researchers are usually interested in a run-length of about 370. The reason for this originated from the fact that if a random variable $Z \sim N(0,1)$ then $P(-3 < Z < 3) = 0.0027$. In other words, if $\beta = 0.0027$, the expected run-length $E(r|\alpha) = \frac{1-0.0027}{0.0027} = 369.37$. However, a $\beta = 0.0027$ will give a much larger average run-length than 370. The reason for this is the variation in α which is illustrated by the posterior distribution in Figure 7. The run-length will also be large if the parameter values of the geometric distributions are small. However, an advantage of the Bayesian procedure is that β can be adjusted so that the average or median run-length takes on a value near 370. In our case, a median run-length of 354 will be used. In Table 9, the average and median run-lengths for $\hat{\alpha}_f$ are given for the different

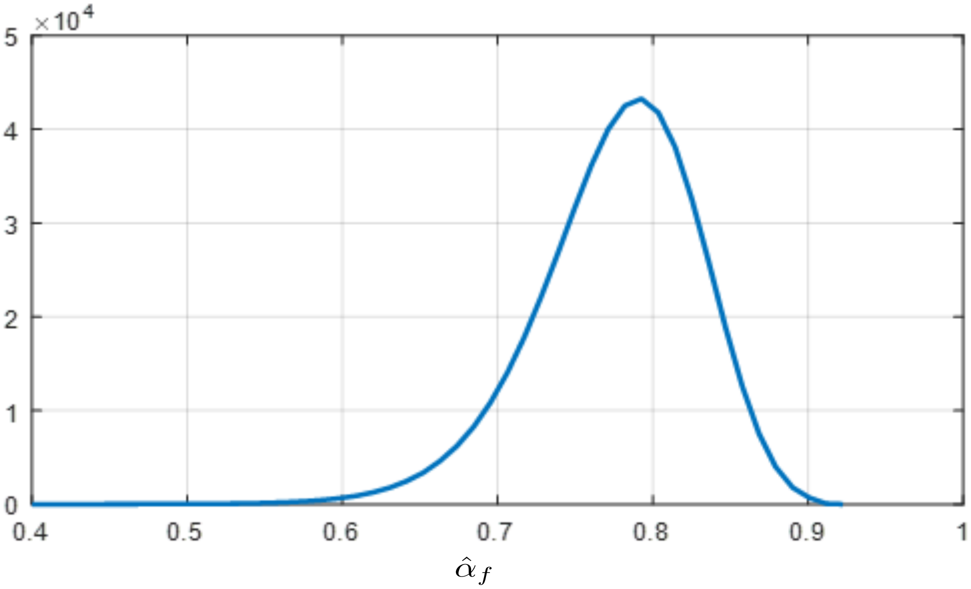

Figure 8. Predictive Density Function $f\left(\hat{\alpha}_f|\hat{\alpha}\right)$ with $\tilde{v}_1 = 360$, $\tilde{v}_2 = 89$ and $\hat{\alpha} = 0.7825$.

values of β. The theoretical run-length is $\frac{1-\beta}{\beta}$.

6. The out of control situation

Assume in a future experiment with $\tilde{I} = 90$ and $J = 5$ the true parameter value of α has changed from 0.78 to 0.74 then the average run-length might change dramatically. If the process is in control and if the desired median run-length is 354, then $\beta = 0.007$ and $R(\beta)$ presents those values of $\hat{\alpha}_f$ that are smaller than $A = 0.6003$ and larger than $B = 0.88$. For the out of control situation the parameter of the geometric distribution is

$$\Psi(\alpha) = \int_{R(\beta)} f\left(\hat{\alpha}_f\,|\alpha\right) d\hat{\alpha}_f = 0.0077,$$

which means that for $\alpha = 0.74$ the average run-length is now

$$E(r|\alpha) = \frac{1-\Psi(\alpha)}{\Psi(\alpha)} = \frac{1-0.0077}{0.0077} = 128.87$$

and the variance of the run-length is

$$Var(r|\alpha) = \frac{1-\Psi(\alpha)}{\Psi^2(\alpha)} = 16\,736.38.$$

A change in run-length from 354 to 129 is quite large and indicates that the process is out of control. In Figure 9, the "in control" and "out of control" situations are illustrated.

Table 9. Run-lengths (r) for $\hat{\alpha}_f$ in the case of different β values. $v_1 = 480$, $v_2 = 119$, $\tilde{v}_1 = 360$, $\tilde{v}_2 = 89$ and $\hat{\alpha} = 0.7825$.

β	$E[r\|\hat{\alpha}]$	$Median[r\|\hat{\alpha}]$	Theoretical Run-Length
0.0050	1754.9	562	199.0
0.0060	1356.5	456	165.7
0.0070	1027.7	354	141.9
0.0080	843.7	305	124.0
0.0090	891.2	250	110.1
0.0100	577.8	215	99.0
0.0150	305.3	120	65.7
0.0200	194.2	80	49.0
0.0250	137.9	58	39.0
0.0300	103.3	44	32.3
0.0350	80.9	35	27.6
0.0400	66.7	30	24.0
0.0450	56.7	25	21.2
0.0500	47.5	21	19.0
0.0550	41.2	19	17.2
0.0600	36.4	17	15.7
0.0650	31.8	14	14.4
0.0700	28.5	13	13.3
0.0750	26.0	12	12.3
0.0800	23.6	11	11.5
0.0850	21.4	10	10.8
0.0900	19.8	9	10.0
0.0950	18.3	8	9.5
0.1000	16.6	7	9.0

7. Example: Bore diameter data

The data in this example are the data from the manufacturing industry provided by Wooluru et al. (2014). The critical quality characteristic is the "Bore diameter" on the driver gear. The number of batches (samples) is $I = 20$ and the number of observations per sample is $J = 5$. The sample Cronbach's alpha is $\hat{\alpha} = 0.4952$. In Figure 10, the posterior density function of α is given and in Figure 11, the predictive density function of $\hat{\alpha}_f$ is given. The number of batches in the future data set are also 20. Therefore, $\tilde{I} = 20$ and $J = 5$.

In Table 12, the average and median run-lengths for $\hat{\alpha}_f$ are given for the different values of β. From the table, it can be seen that for an average run-length in the vicinity of 370, $\beta = 0.0180$ should be used instead of $\beta = 0.0027$.

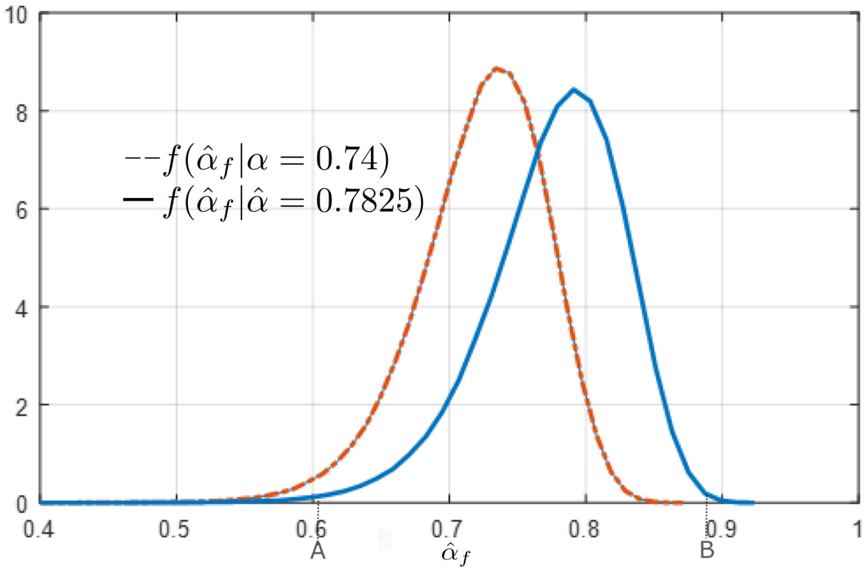

Figure 9. Graphs of the in control and out of control situations.

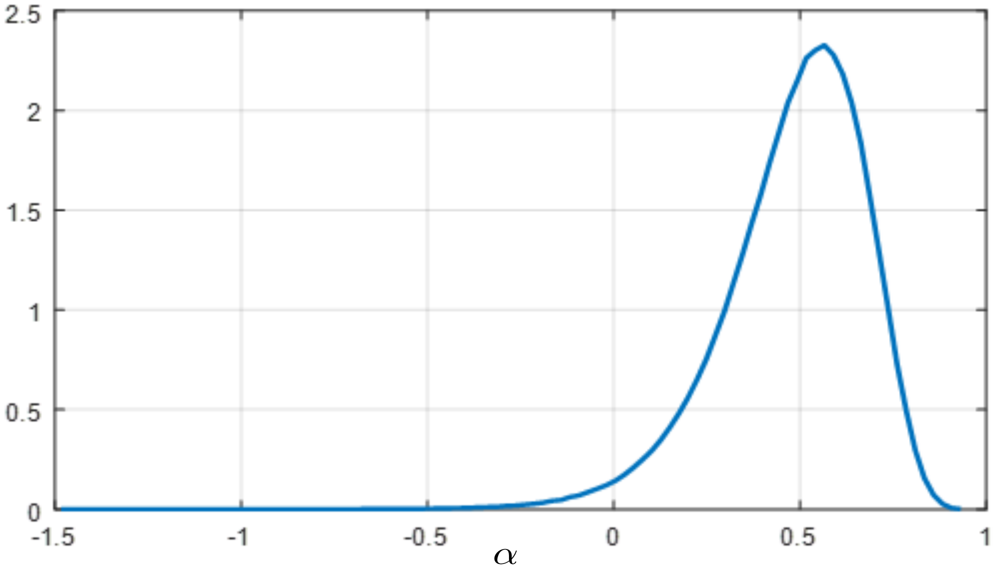

Figure 10. Posterior distribution $f(\alpha|\hat{\alpha})$, $v_1 = 80$ and $v_2 = 19$.

Table 10. Summary statistics and credibility intervals of α.

Mean (α)	Median (α)	Mode (α)	Var (α)
0.4822	0.5089	0.56	0.0359
90% Equal-tailed Interval		95% Equal-tailed Interval	
$(0.1330; 0.7414)$		$(0.0400; 0.7741)$	

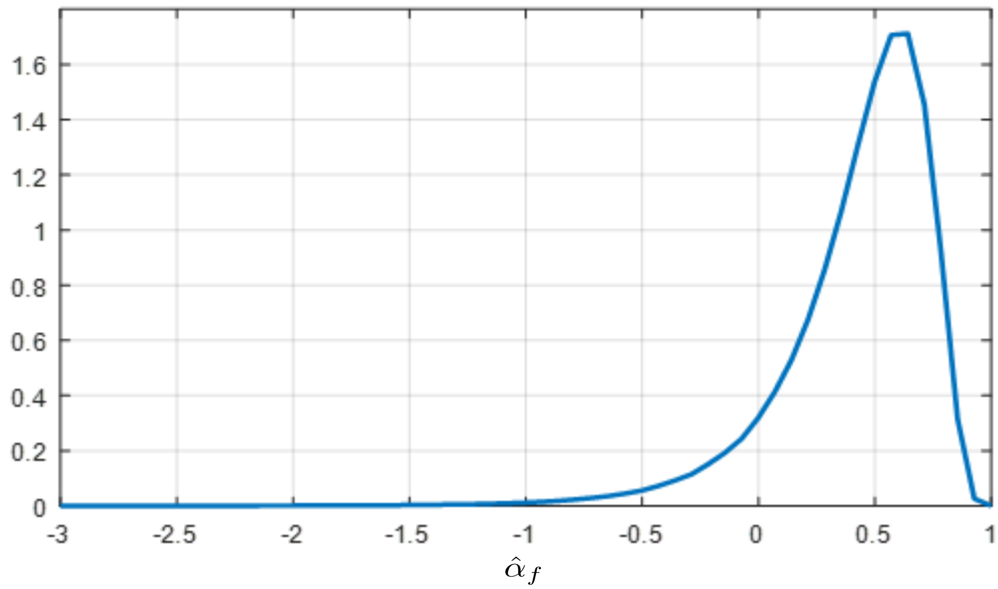

Figure 11. Predictive density $f\left(\hat{\alpha}_f|\hat{\alpha}\right)$, $\hat{v}_1 = 80$ and $\hat{v}_2 = 19$.

8. Conclusion

In this paper, statistical process control limits have been obtained for Cronbach's coefficient alpha in the case of the balanced one-way random effects model. This has been achieved by deriving the predictive distribution of a future (unseen) Cronbach's coefficient alpha . For given variance components, it was shown that the predictive density function of $f\left(\hat{\alpha}_f|\alpha\right)$ can be derived analytically. The Jeffreys independence prior was used to derive the posterior distribution of α. The unconditional posterior predictive density function $f\left(\hat{\alpha}_f|\hat{\alpha}\right)$ can be obtained by Monte Carlo simulation or numerical integration. The predictive density function $f\left(\hat{\alpha}_f|\hat{\alpha}\right)$ as well as the conditional predictive density functions $f\left(\hat{\alpha}_f|\alpha\right)$ can be used to determine the run-length and the average run-length. The distribution of the run-length $f\left(r|\hat{\alpha}\right)$ is the average of a large number of geometric distributions each with its own parameter value. Three examples were considered. The first example had to do with Dyestuff data from Box and Tiao (1973). In the second example, it is assumed that the number of Dyestuff samples has increased from six to 120, and the number of samples in a future (unseen) data set is ninety. The third example is from Wooluru et al. (2014) and is measured values of "Box diameter" on the driver gear. The results showed that the average and median run-lengths are usually larger than the theoretical values. An advantage of the Bayesian procedure, however, is that control

Table 11. Summary statistics and credibility intervals of $\hat{\alpha}_f$ for the Bore diameter data, $\beta = 0.1$.

Mean $(\hat{\alpha}_f)$	Median $(\hat{\alpha}_f)$	Mode $(\hat{\alpha}_f)$	Var $(\hat{\alpha}_f)$
0.4213	0.4951	0.61	0.1064
90% Equal tail Interval		95% HPD Interval	
$(-0.1901; 0.7858)$		$(-0.2091; 0.8819)$	

Table 12. Run-lengths (r) for $\hat{\alpha}_f$ in the case of different β values. $v_1 = 80$, $v_2 = 19$, $\tilde{v}_1 = 80$, $\tilde{v}_2 = 19$.

β	$E[r\|\hat{\alpha}]$	$Median[r\|\hat{\alpha}]$
0.0050	3030.3	820
0.0060	2355.4	643
0.0070	1887.2	528
0.0080	1493.5	431
0.0090	1214.6	353
0.0100	1023.9	304
0.0110	870.8	266
0.0120	733.8	231
0.0130	638.7	203
0.0140	564.6	185
0.0150	504.9	165
0.0160	453.7	151
0.0170	406.5	138
0.0180	371.7	127
0.0190	336.5	115
0.0200	305.9	106

limits can be adjusted in such a way that the average or median run-length has a specific value.

Appendix A: Proof of Theorem 1

From the posterior distribution it follows that

$$\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + J\sigma_r^2} \sim \frac{\frac{v_1 m_1}{\chi_{v_1}^2}}{\frac{v_2 m_2}{\chi_{v_2}^2}} = \frac{m_1}{m_2} \frac{\frac{\chi_{v_2}^2}{v_2}}{\frac{\chi_{v_1}^2}{v_1}}.$$

Therefore

$$\frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + J\sigma_r^2} \sim (1 - \hat{\alpha}) F_{v_2, v_1}$$

and

$$1 - \alpha \sim (1 - \hat{\alpha}) F_{v_2, v_1}.$$

Therefore

$$\alpha|\hat{\alpha} \sim 1-(1-\hat{\alpha})\, F_{v_2,v_1}.$$

Let $f = F_{v_2,v_1}$, then

$$g(f) = K_1 \left(\frac{v_2}{v_1}\right)^{\frac{1}{2}v_2} f^{\frac{1}{2}v_2-1}\left(1+\frac{v_2}{v_1}f\right)^{-\frac{1}{2}(v_1+v_2)}.$$

Since $\alpha = 1-(1-\hat{\alpha})\, f$, it follows that

$$f = \frac{1-\alpha}{1-\hat{\alpha}} \text{ and } \left|\frac{df}{d\alpha}\right| = \frac{1}{1-\hat{\alpha}}.$$

Therefore

$$p\left(\alpha|\hat{\alpha}\right) = K_1\left(\frac{v_2}{v_1}\right)^{\frac{1}{2}v_2}\left(\frac{1}{1-\hat{\alpha}}\right)^{\frac{1}{2}v_2}(1-\alpha)^{\frac{1}{2}v_2-1}\times\left[1+\frac{v_2}{v_1}\left(\frac{1-\alpha}{1-\hat{\alpha}}\right)\right]^{-\frac{1}{2}(v_1+v_2)}$$

where $K_1 = \frac{\Gamma\left(\frac{v_1+v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)}$. Since

$$\begin{aligned} E\left(\alpha|\hat{\alpha}\right) &= 1-(1-\hat{\alpha})\, E\left(F_{v_2,v_1}\right) \\ &= 1-(1-\hat{\alpha})\left(\frac{v_1}{v_1-2}\right) \end{aligned}$$

and

$$\begin{aligned} Var\left(\alpha|\hat{\alpha}\right) &= (1-\hat{\alpha})^2\, Var\left(F_{v_2,v_1}\right) \\ &= (1-\hat{\alpha})^2\,\frac{2v_1^2\left(v_2+v_1-2\right)}{v_2\left(v_1-2\right)^2\left(v_1-4\right)}. \end{aligned}$$

Appendix B: Proof of Theorem 2

Let $\tilde{f} = F_{\tilde{v}_1,\tilde{v}_2}$. Therefore

$$f(\tilde{f}) = K_2\left(\frac{\tilde{v}_1}{\tilde{v}_2}\right)^{\frac{1}{2}\tilde{v}_1}\tilde{f}^{\frac{1}{2}\tilde{v}_1-1}\left(1+\frac{\tilde{v}_1}{\tilde{v}_2}\tilde{f}\right)^{-\frac{1}{2}(\tilde{v}_1+\tilde{v}_2)}.$$

Since $\hat{\alpha}_f = 1-(1-\alpha)\,\tilde{f}$, it follows that

$$\tilde{f} = \frac{1-\hat{\alpha}_f}{1-\alpha} \quad \text{and} \quad \left|\frac{d\tilde{f}}{d\hat{\alpha}_f}\right| = \frac{1}{1-\alpha}.$$

Therefore

$$\begin{aligned} f(\hat{\alpha}_f|\alpha) &= K_2\left(\frac{\tilde{v}_1}{\tilde{v}_2}\right)^{\frac{1}{2}\tilde{v}_1}\left(\frac{1-\hat{\alpha}_f}{1-\alpha}\right)^{\frac{1}{2}\tilde{v}_1-1}\left[1+\frac{\tilde{v}_1}{\tilde{v}_2}\left(\frac{1-\hat{\alpha}_f}{1-\alpha}\right)\right]^{-\frac{1}{2}(\tilde{v}_1+\tilde{v}_2)}\left(\frac{1}{1-\alpha}\right) \\ &= K_2\left(\frac{\tilde{v}_1}{\tilde{v}_2}\right)^{\frac{1}{2}\tilde{v}_1}\left(\frac{1}{1-\alpha}\right)^{\frac{1}{2}\tilde{v}_1}\left(1-\hat{\alpha}_f\right)^{\frac{1}{2}\tilde{v}_1-1}\left[1+\frac{\tilde{v}_1}{\tilde{v}_2}\left(\frac{1-\hat{\alpha}_f}{1-\alpha}\right)\right]^{-\frac{1}{2}(\tilde{v}_1+\tilde{v}_2)}, \end{aligned}$$

where $K_2 = \frac{\Gamma\left(\frac{\tilde{v}_1+\tilde{v}_2}{2}\right)}{\Gamma\left(\frac{\tilde{v}_1}{2}\right)\Gamma\left(\frac{\tilde{v}_2}{2}\right)}$. Also

$$\begin{aligned} E\left(\hat{\alpha}_f|\alpha\right) &= 1-(1-\alpha)\,E\left(F_{\tilde{v}_1,\tilde{v}_2}\right) \\ &= 1-(1-\alpha)\left(\frac{\tilde{v}_2}{\tilde{v}_2-2}\right) \end{aligned}$$

and

$$\begin{aligned} Var\left(\hat{\alpha}_f|\hat{\alpha}\right) &= (1-\alpha)^2\,Var\left(F_{\tilde{v}_1,\tilde{v}_2}\right) \\ &= (1-\alpha)^2\,\frac{2\left(\tilde{v}_2\right)^2\left(\tilde{v}_2+\tilde{v}_1-2\right)}{\tilde{v}_1\left(\tilde{v}_2-2\right)^2\left(\tilde{v}_2-4\right)}. \end{aligned}$$

Appendix C: Proof of Theorem 3

$$\hat{\alpha}_f|\alpha = 1-(1-\alpha)\,F_{\tilde{v}_1,\tilde{v}_2} \tag{13}$$

and

$$\alpha|\hat{\alpha} = 1-(1-\hat{\alpha})\,F_{v_2,v_1} \tag{14}$$

From Equation 13, it follows that

$$\begin{aligned} E\left(\hat{\alpha}_f|\alpha\right) &= 1-(1-\alpha)\,E\left(F_{\tilde{v}_1,\tilde{v}_2}\right) \\ &= 1-(1-\alpha)\left(\frac{\tilde{v}_2}{\tilde{v}_2-2}\right) \end{aligned}$$

and from Equation 14, it follows that

$$(1-\alpha)\sim(1-\hat{\alpha})\,F_{v_2,v_1}.$$

Therefore

$$\begin{aligned} E\left(\hat{\alpha}_f|\hat{\alpha}\right) &= E\left\{E\left(\hat{\alpha}_f|\alpha\right)\right\} \\ &= 1-(1-\hat{\alpha})\left(\frac{v_1}{v_1-2}\right)\left(\frac{\tilde{v}_2}{\tilde{v}_2-2}\right). \end{aligned}$$

Also, for the variance,

$$Var\left(\hat{\alpha}_f|\hat{\alpha}\right) = E\left\{Var\left(\hat{\alpha}_f|\alpha\right)\right\} + Var\left\{E\left(\hat{\alpha}_f|\alpha\right)\right\}.$$

Now,

$$Var\left(\hat{\alpha}_f|\alpha\right) = (1-\alpha)^2\,Var\left(F_{\tilde{v}_1,\tilde{v}_2}\right)$$

and

$$E\left(\hat{\alpha}_f|\alpha\right) = 1-(1-\alpha)\left(\frac{\tilde{v}_2}{\tilde{v}_2-2}\right).$$

Further,

$$\begin{aligned} E\left\{Var\left(\hat{\alpha}_f|\alpha\right)\right\} =& E\left(1-\alpha\right)^2 Var\left(F_{\tilde{v}_1,\tilde{v}_2}\right) \\ =& \left\{Var\left(1-\alpha\right)+\left[E\left(1-\alpha\right)\right]^2\right\} Var\left(F_{\tilde{v}_1,\tilde{v}_2}\right). \end{aligned}$$

Since

$$(1-\alpha) \sim (1-\hat{\alpha}) F_{v_2,v_1},$$

it follows that

$$Var\left(1-\alpha\right) = \left(1-\hat{\alpha}\right)^2 Var\left(F_{v_2,v_1}\right)$$

and

$$\left[E\left(1-\alpha\right)\right]^2 = \left(1-\hat{\alpha}\right)^2 \left(\frac{v_1}{v_1-2}\right)^2.$$

Therefore,

$$\begin{aligned} & E\left\{Var\left(\hat{\alpha}_f|\alpha\right)\right\} \\ &= \left(1-\hat{\alpha}\right)^2 \left\{Var\left(F_{v_2,v_1}\right) + \left(\frac{v_1}{v_1-2}\right)^2\right\} Var\left(F_{\tilde{v}_1,\tilde{v}_2}\right) \end{aligned}$$

and

$$\begin{aligned} Var\left\{E\left(\hat{\alpha}_f|\alpha\right)\right\} &= Var\left\{1-\left(1-\alpha\right)\left(\frac{\tilde{v}_2}{\tilde{v}_2-2}\right)\right\} \\ &= \left(\frac{\tilde{v}_2}{\tilde{v}_2-2}\right)^2 \left(1-\hat{\alpha}\right)^2 Var\left(F_{v_2,v_1}\right). \end{aligned}$$

From this it follows that

$$\begin{aligned} Var\left(\hat{\alpha}_f|\hat{\alpha}\right) = \left(1-\hat{\alpha}\right)^2 &\left\{Var\left(F_{v_2,v_1}\right) + \left(\frac{v_1}{v_1-2}\right)^2\right\} Var\left(F_{\tilde{v}_1,\tilde{v}_2}\right) \\ &+ \left(1-\hat{\alpha}\right)^2 \left(\frac{\tilde{v}_2}{\tilde{v}_2-2}\right)^2 Var\left(F_{v_2,v_1}\right). \end{aligned}$$

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