South African Statistical Journal
Vol. 58, No. 2, pp. 81–98
https://doi.org/10.37920/sasj.2024.58.2.1

A general multiple-repair-attempt process and its application to optimal age replacement

Amy Langston[1] *and Maxim Finkelstein*[2,3]

[1]Department of Statistics, Rhodes University, Makhanda, South Africa
[2]Department of Mathematical Statistics, University of the Free State, Bloemfontein, South Africa
[3]Department of Management Science, University of Strathclyde, Glasgow, UK

Considerable attention in reliability literature has been given to studying various repair models. These models are often described under the assumption of minimal repair. However, repairs of a failed system can be unsuccessful and, therefore, should be repeated until a successful attempt. At certain instances, the assumption of minimal repairs is too restrictive, as the repair can be worse than minimal due to adverse effects of the previous repair attempts, external and internal shocks to a system, insufficient quality of repair, etc. Therefore, in this paper, we use the generalised Polya process to define and derive useful properties for a more general multiple-repair-attempt process with the worse than minimal repair. This has the minimal repair model as a special case. As an application, the corresponding age replacement policy for the general multiple-repair-attempt model is defined and the optimal solutions are obtained. Detailed numerical illustrations support our findings.

Keywords: Generalised Polya process, Maintenance, Multiple repair attempts, Optimal age replacement, Worse-than-minimal repair.

1. Introduction

Repair models are extensively studied in reliability literature. These models provide a mechanism to analyse the performance of repairable systems and also define the corresponding maintenance policies. Perfect repairs bring a system after repair to 'as good as new state' (modelled by the classical renewal process; Barlow and Proschan, 1975). However, in practice, most repairs are imperfect. The most commonly used imperfect repair model is described by minimal repair. Minimal repair restores a system to the 'as bad as old' state and is considered to be a fairly reasonable assumption in reality. The concept of minimal repair was introduced by Barlow and Hunter (1960), and it is well-known that the the process of instantaneous (repair time is negligible) minimal repairs can be described by the non-homogeneous Poisson process (NHPP) with the rate given by the failure rate that corresponds to the underlying lifetime distribution. The NHPP often provides the closed-form analytical results and easy derivations of mathematical properties, and as such, has allowed for the comprehensive study of various preventative maintenance models under the assumption of minimal repairs; see, for

Corresponding author: Amy Langston (a.langston@ru.ac.za)
MSC2020 subject classifications: 60G55, 90B25, 60K10

example, Nakagawa (1980), Beichelt and Fischer (1980), Nakagawa (1986), Park et al. (2000), Tadj et al. (2011), Badía et al. (2020), Liu and Wang (2021), to name a few.

A worse-than-minimal repair restores a system to a 'worse than old' state and considers the impact that previous failures/repairs have had on the overall state of the system. For example, when a component of a system fails, it may cause additional stress or damage to the non-failed components of the system, resulting in system degradation. This would increase the failure rate function of the system, and so the system will be 'worse off' than it was just before failure. As shown in Lee and Cha (2016), worse-than-minimal repair can be described by the generalised Polya process (GPP), which was introduced and extensively studied in Cha (2014). Although more realistic in some instances, this type of repair has not been as extensively studied in the literature; see, for example, Badía et al. (2018), Cha and Finkelstein (2018), and Cha et al. (2023) for some notable considerations.

Cha and Finkelstein (2023) have considered recently a multi-attempt minimal repair model where an attempt to repair a system after failure can be unsuccessful and, therefore, should be repeated until a successful minimal repair. This repair process was described using the non-stationary/non-homogeneous Polya-Aeppli process (see Chukova and Minkova (2019) for the generalisation of the Polya-Aeppli process considered in Minkova (2004)), where each repair attempt is successful with probability ρ (resulting in a minimal repair), and unsuccessful with probability $1-\rho$. An application of this model to optimal replacement was considered here under a periodic replacement policy, where an item is replaced on reaching age T (see Nakagawa, 2005). Cha and Finkelstein (2024) provided a further generalisation to the application of the model in Cha and Finkelstein (2023) by considering a practical extension to the preventative maintenance policy, where an item is replaced at some specified age T or after $k \geq 1$ consecutive unsuccessful minimal repairs, whichever comes first.

An unsuccessful repair attempt can occur for a number of reasons. For example, (a) complex systems with intricate components (such as software, advanced electronic devices, and machinery) may require multiple repair attempts due to the complexity of the internal structure, (b) systems that contain a number of interconnected parts (such as automotive and medical systems) may require multiple attempts to address the cascading effects of the initial failure, (c) technicians incorrectly diagnosing the root cause of a failure may result in multiple repair attempts to correctly identify and address the underlying issue (this example could also be linked to a trial-and-error repair strategy if the exact cause is unknown), and (d) multiple repair attempts may be necessary if the spare parts used in the initial attempts are faulty (or of lower quality) and do not immediately resolve the issue. This makes the possibility of unsuccessful repair attempts a practically important consideration when trying to model real-life maintenance for various systems.

The multi-attempt model in Cha and Finkelstein (2023), and hence Cha and Finkelstein (2024), like many others, is built under the assumption that the successful repairs are minimal. This assumption is not unrealistic, however, it can still be too restrictive to fully describe the underlying process. In fact, at some instances, it may be more realistic to consider the successful repairs as worse-than-minimal due to the adverse effects of the previous repair attempts, external and internal shocks to the system, insufficient quality of a successful repair attempt, and so on. As a more specific example, consider example (a) mentioned earlier. It is likely that when a system with a complex internal structure is repaired, some components get damaged during the repair operation or simply due to the additional stress on these components during the failure. Therefore, the repair does not restore the system to as it was before the failure, but rather to a 'weakened' state. Even in example (d), replacing a part with a lower-quality part may return the system to an operable state after multiple repair attempts, but the system will be more susceptible to failure in the future. That is, leaving the system 'worse off' than before the initial failure. Therefore, in this paper, we consider a more general multiple-repair-attempt process to account for this practical possibility. A new methodology had to be developed, as the non-homogeneous Polya-Aeppli process can no longer be used as in the case of minimal repairs. Instead, as the GPP describes repairs that are worse-than-minimal, we use this as the underlying assumption for the successful repair attempts to define our general model. Further, appropriate analytical properties are derived under this assumption which allows for consideration of the corresponding optimal preventative maintenance problem. It will also be shown that our general multiple-repair-attempt process has the multi-attempt minimal repair model, described in Cha and Finkelstein (2023), as its special case.

This paper is organised as follows. In Section 2, we provide some preliminary definitions and properties for the GPP, which will be necessary for the forthcoming discussions. In Section 3, the multiple-repair-attempt process is defined in the context of worse-than-minimal repairs. Section 4 details a further generalisation of the described multiple-repair-attempt process to the case where the probability of a successful repair is time-dependent. In Section 5, we consider the application of this process to the optimal age replacement problem and provide the corresponding numerical illustrations. Concluding remarks are given in Section 6.

2. Preliminaries

This paper will define a multiple-repair-attempt process with the underlying assumption that the successful repair attempts are modelled by the GPP. Therefore, in this section, we will provide a brief yet sufficient discussion of the GPP and the relevant properties that will be necessary for developing the corresponding counting process.

The GPP was characterised in Cha (2014) through the notion of stochastic intensity. Stochastic intensity (or the intensity process), λ_t, $t \geq 0$ is a method to mathematically describe point processes (see Aven and Jensen, 1999, 2000). For an orderly (or simple) point process $\{M(t), t \geq 0\}$ whose history (internal filtration) in $[0, t)$ is $H_{t-} \equiv \{M(u), 0 \leq u < t\}$, the stochastic intensity λ_t is defined through the following limit

$$\lambda_t = \lim_{\Delta t \to 0} \frac{P\left(M\left(t, t+\Delta t\right) = 1 \mid H_{t-}\right)}{\Delta t} = \lim_{\Delta t \to 0} \frac{E\left[M\left(t, t+\Delta t\right) \mid H_{t-}\right]}{\Delta t}, \quad (1)$$

where $M(t, t+\Delta t)$ represents the number of events in $[t, t+\Delta t)$ (see, for example Finkelstein, 2008; Finkelstein and Cha, 2013; Cha and Finkelstein, 2018). Note that the history of the process H_{t-} is the set of all point events in $[0, t)$, and can be equivalently defined in terms of $M(t-)$ and the sequential arrival times of the events $0 \leq T_1 \leq T_2 \leq \ldots \leq T_{M(t-)} < t$, where T_i is the time from 0 until the arrival of the ith event in $[0, t)$.

The heuristic interpretation of the stochastic intensity defined in (1) is given by $\lambda_t \mathrm{d}t = E[\mathrm{d}M(t) \mid H_{t-}]$. This is very similar to that of the ordinary failure rate of a random variable (Aven and Jensen, 1999). It is well known that the NHPP with rate $\lambda(t)$ can be characterised through the deterministic stochastic intensity. That is, $\lambda_t = \lambda(t)$, $t \geq 0$.

In Cha (2014), the GPP is defined as follows:

Definition 1 (Generalised Polya process)**.** A counting process $\{M(t), t \geq 0\}$ is said to be a generalised Polya process (GPP) with the set of parameters $(\lambda(t), \alpha, \beta)$, $\alpha \geq 0, \beta > 0$, if

i. $M(0) = 0$;

ii. $\lambda_t = (\alpha M(t-) + \beta)\, \lambda(t)$.

From Cha (2014), the GPP with the set of parameters $(\lambda(t), \alpha = 0, \beta = 1)$ reduces to the NHPP with rate $\lambda(t)$. Therefore, the GPP can be understood as a more general counting process, with the NHPP as its special case. From the definition of the GPP, it is also clear that α describes how much influence the number of prior events (the history) has on the probability of occurrence of an event in an infinitesimal interval of time, whereas $\lambda(t)$ can be interpreted as some baseline rate function for the process.

The corresponding failure/repair process for the GPP was defined in Cha (2014) for the parameter set $(\lambda(t), \alpha, \beta)$ under the assumption that the duration of the repair is negligible. Although already mathematically tractable, a more suitable reparameterisation was presented in Lee and Cha (2016) to define the GPP repair process without loss of generality. The definition of this GPP repair process is given in Definition 2.

Definition 2 (GPP repair)**.** For a system with failure rate $\lambda(t)$, a repair is said to be a 'GPP repair' with parameter α if $\{M(t), t \geq 0\}$ is the GPP with the parameter set $(\lambda(t), \alpha, 1)$.

It obviously follows that the stochastic intensity corresponding to this GPP repair process is

$$\lambda_t = (\alpha M(t-) + 1)\, \lambda(t)$$

and is equivalent to the stochastic intensity in Definition 1 with $\beta = 1$. Throughout this paper, we will use the reparametrisation provided through Definition 2 as it sufficiently describes repairs that are worse-than-minimal. For ease of notation, we will refer to the GPP with parameter set $(\lambda(t), \alpha)$.

In order to derive the distribution of the multiple-repair-attempt process and define various other properties, we will require the following supplementary results:

Lemma 1. For the GPP with set of parameters $(\lambda(t), \alpha)$, $\alpha > 0$, the following properties hold:

i. The distribution of $M(t)$ is given by

$$P(M(t) = m) = \frac{\Gamma\left(\frac{1}{\alpha} + m\right)}{\Gamma\left(\frac{1}{\alpha}\right) m!} \left(1 - \exp\{-\alpha\Lambda(t)\}\right)^m \left(\exp\{-\Lambda(t)\}\right), \quad m = 0, 1, 2, \ldots,$$

where $\Lambda(t) \equiv \int_0^t \lambda(u)\mathrm{d}u$.

ii. $E[M(t)] = \frac{1}{\alpha}(\exp\{\alpha\Lambda(t)\} - 1)$.

iii. The conditional joint distribution of the arrival times $(T_1, T_2, \ldots, T_{M(t)})$, given that $M(t) = m$, is

$$f_{(T_1, T_2, \ldots, T_{M(t)} | M(t))}\left(T_1, T_2, \ldots, T_{M(t)} \mid m\right) = m! \prod_{i=1}^{m} \frac{\alpha\lambda(t_i)\exp\{\alpha\Lambda(t_i)\}}{\exp\{\alpha\Lambda(t)\} - 1},$$

for $0 \leq t_1 \leq t_2 \leq \ldots \leq t_m$.

The proofs for these properties, as well as an extensive discussion of other properties for the GPP with parameter set $(\lambda(t), \alpha, \beta)$, $\alpha \geq 0, \beta > 0$ can be found in Cha (2014).

3. The multiple-repair-attempt process

Consider a system with a lifetime that is characterised by the absolutely continuous cdf $F(t)$, the survival function $\bar{F}(t) = 1 - F(t)$, and failure rate $\lambda(t)$. When the system fails, a repair action is performed. This repair action can be unsuccessful and, therefore, should be repeated until a successful attempt. First, for simplicity, consider the case where each repair attempt is either unsuccessful with probability $(1 - \rho)$ or successful with probability ρ. The successful repair attempt is considered to be a worse-than-minimal repair of the system.

Let $M(t)$ represent the number of system failures in the interval $(0, t]$. Assuming the time taken to successfully repair the system after a failure is negligible, $M(t)$ can also be interpreted as the number of successful worse-than-minimal repairs by time t. Then, obviously, the counting process $\{M(t), t \geq 0\}$ is the GPP with parameter set $(\lambda(t), \alpha)$. Let $N(t)$ represent the total number of repair attempts, both successful and unsuccessful, in the interval $(0, t]$. If we consider $\{N(t), t \geq 0\}$ as a counting process and let $X_1, X_2, \ldots$ be iid geometric random variables with parameter ρ and independent of $M(t)$, then $N(t)$ and the corresponding multiple-repair-attempt process can be defined as follows:

Definition 3. Let $\{M(t), t \geq 0\}$ be the GPP with parameter set $(\lambda(t), \alpha)$ and define $N(t)$ as

$$N(t) = X_1 + X_2 + \cdots + X_{M(t)},$$

where $X_1, X_2, \ldots$ are iid geometric random variables with parameter ρ and independent of $M(t)$. Then, the counting process $\{N(t), t \geq 0\}$ can be defined as the multiple-repair-attempt process with parameter set $(\lambda(t), \alpha, \rho)$.

In what follows, we will derive the distribution of $N(t)$ and present various properties to further describe the process.

Proposition 3.1. The probability generating function (pgf) for $N(t)$ is given by

$$\psi_{N(t)}(s) = \frac{1}{\left(\exp\{\alpha\Lambda(t)\} + \psi_X(s)\left(1 - \exp\{\alpha\Lambda(t)\}\right)\right)^{\frac{1}{\alpha}}},$$

where $\psi_X(s) = s\rho/(1 - (1 - \rho)s)$ is the pgf of a geometric random variable with parameter ρ.

Proof. By definition,

$$\psi_{N(t)}(s) = E\left[s^{X_1+X_2+\cdots+X_{M(t)}}\right] = E\left[E\left[s^{\sum_{i=1}^{M(t)} X_i} \mid M(t)\right]\right].$$

Consider,

$$E\left[s^{\sum_{i=1}^{M(t)} X_i} \mid M(t) = m\right] = E\left[s^{\sum_{i=1}^{m} X_i}\right] = \prod_{i=1}^{m} E\left(s^{X_i}\right) = (\psi_X(s))^m,$$

where the last equality holds since $X_1, X_2, \ldots$ are iid geometric random variables with parameter ρ

and common pgf $\psi_X(s) = s\rho/(1-(1-\rho)s)$. Therefore,

$$\psi_{N(t)}(s) = \sum_{m=0}^{\infty} (\psi_X(s))^m \frac{\Gamma\left(\frac{1}{\alpha}+m\right)}{\Gamma\left(\frac{1}{\alpha}\right)m!} (1-\exp\{-\alpha\Lambda(t)\})^m (\exp\{-\Lambda(t)\})$$
$$= \frac{1}{(\exp\{\alpha\Lambda(t)\} + \psi_X(s)(1-\exp\{\alpha\Lambda(t)\}))^{\frac{1}{\alpha}}}. \quad \blacksquare$$

Proposition 3.2. The distribution of $N(t)$ is given by

$$P(N(t)=0) = \exp\{-\Lambda(t)\},$$
$$P(N(t)=n) = \sum_{m=1}^{n} \binom{n-1}{m-1} \rho^m (1-\rho)^{n-m} \frac{\Gamma\left(\frac{1}{\alpha}+m\right)}{\Gamma\left(\frac{1}{\alpha}\right)m!} (1-\exp\{-\alpha\Lambda(t)\})^m (\exp\{-\Lambda(t)\}),$$

for $n = 1, 2, \ldots$.

Proof. Recall that $N(t)$ can be expressed as

$$N(t) = X_1 + X_2 + \cdots + X_{M(t)},$$

where $X_1, X_2, \ldots$ are iid geometric random variables with parameter ρ and independent of $M(t)$. Then, obviously, for $n = 0$,

$$P(N(t)=0) = P(M(t)=0) = \exp\{-\Lambda(t)\}.$$

For $n = 1, 2, \ldots$,

$$P(N(t)=n) = \sum_{m=1}^{n} P(N(t)=n \mid M(t)=m)\, P(M(t)=m),$$

where $(N(t)=n \mid M(t)=m)$ follows a negative binomial random variable with parameters (ρ, m). Thus,

$$P(N(t)=n \mid M(t)=m) = \binom{n-1}{m-1} \rho^m (1-\rho)^{n-m},$$

and, for $n = 1, 2, \ldots$,

$$P(N(t)=n) = \sum_{m=1}^{n} \binom{n-1}{m-1} \rho^m (1-\rho)^{n-m} \frac{\Gamma\left(\frac{1}{\alpha}+m\right)}{\Gamma\left(\frac{1}{\alpha}\right)m!} (1-\exp\{-\alpha\Lambda(t)\})^m (\exp\{-\Lambda(t)\}). \quad \blacksquare$$

It can also be shown that the following relationships hold for the interval $(s, s+t]$:

$$P(N(s+t)-N(s)=0) = \left(\frac{\exp\{-\alpha\Lambda(s+t)\}}{1+\exp\{-\alpha\Lambda(s+t)\} - \exp\{-\alpha[\Lambda(s+t)-\Lambda(s)]\}}\right)^{\frac{1}{\alpha}},$$

and, for $n = 1, 2, \ldots,$

$$P\left(N\left(s+t\right)-N\left(s\right)=n\right)=\sum_{m=1}^{n}\left[\binom{n-1}{m-1}\rho^{m}\left(1-\rho\right)^{n-m}\frac{\Gamma\left(\frac{1}{\alpha}+m\right)}{\Gamma\left(\frac{1}{\alpha}\right)m!}\right.$$
$$\times\left(\frac{1-\exp\left\{-\alpha\left[\Lambda\left(s+t\right)-\Lambda\left(s\right)\right]\right\}}{1+\exp\left\{-\alpha\Lambda\left(s+t\right)\right\}-\exp\left\{-\alpha\left[\Lambda\left(s+t\right)-\Lambda\left(s\right)\right]\right\}}\right)^{m}$$
$$\left.\times\left(\frac{\exp\left\{-\alpha\Lambda\left(s+t\right)\right\}}{1+\exp\left\{-\alpha\Lambda\left(s+t\right)\right\}-\exp\left\{-\alpha\left[\Lambda\left(s+t\right)-\Lambda\left(s\right)\right]\right\}}\right)^{\frac{1}{\alpha}}\right].$$

The proofs follow similarly to Proposition 3.2 and have been omitted for brevity.

Proposition 3.3. The mean and the variance for $N(t)$ are given by

$$E\left[N\left(t\right)\right]=\frac{\exp\left\{\alpha\Lambda\left(t\right)\right\}-1}{\alpha\rho}$$

and

$$Var\left[N\left(t\right)\right]=\frac{\left(\exp\left\{\alpha\Lambda\left(t\right)\right\}-1\right)\left(\exp\left\{\alpha\Lambda\left(t\right)\right\}+\left(1-\rho\right)\right)}{\alpha\rho^{2}},$$

respectively.

Proof. This follows from $E[N(t)] = \psi'_{N(t)}(1)$ and $Var[N(t)] = \psi''_{N(t)}(1)+\psi'_{N(t)}(1)-(\psi'_{N(t)}(1))^2$. ■

Consider now the conditional joint distribution of the inter-arrival times $(T_1, T_2, \ldots, T_{M(t)} \mid N(t) = n, M(t) = m)$. This is another useful result that further describes the properties of the multiple-repair-attempt process. Observe that

$$P\left(N\left(t\right)=n,M\left(t\right)=m\right)$$
$$=\binom{n-1}{m-1}\rho^{m}\left(1-\rho\right)^{n-m}\frac{\Gamma\left(\frac{1}{\alpha}+m\right)}{\Gamma\left(\frac{1}{\alpha}\right)m!}\left(1-\exp\left\{-\alpha\Lambda\left(t\right)\right\}\right)^{m}\left(\exp\left\{-\Lambda\left(t\right)\right\}\right)$$

for $1 \le m \le n, n = 1, 2, \ldots.$

The joint distribution of $(T_1, T_2, \ldots, T_{M(t)}, N(t) = n, M(t) = m)$ is given by

$$f\left(t_{1},t_{2},\ldots,t_{m},n,m\right)$$
$$=\binom{n-1}{m-1}\rho^{m}\left(1-\rho\right)^{n-m}\lambda\left(t_{1}\right)\exp\left\{-\Lambda\left(t_{1}\right)\right\}\left(1+\alpha\right)\lambda\left(t_{2}\right)\exp\left\{-\left(1+\alpha\right)\left[\Lambda\left(t_{2}\right)-\Lambda\left(t_{1}\right)\right]\right\}$$
$$\times\cdots\times\left(1+\left(m-1\right)\alpha\right)\lambda\left(t_{m}\right)\exp\left\{-\left(1+\left(m-1\right)\alpha\right)\left[\Lambda\left(t_{m}\right)-\Lambda\left(t_{m-1}\right)\right]\right\}$$
$$\times\exp\left\{-\left(1+m\alpha\right)\left[\Lambda\left(t\right)-\Lambda\left(t_{m}\right)\right]\right\}$$
$$=\binom{n-1}{m-1}\rho^{m}\left(1-\rho\right)^{n-m}\frac{\Gamma\left(\frac{1}{\alpha}+m\right)}{\Gamma\left(\frac{1}{\alpha}\right)}\left(\prod_{i=1}^{m}\alpha\lambda\left(t_{i}\right)\exp\left\{\alpha\Lambda\left(t_{i}\right)\right\}\right)\exp\left\{-\left(1+m\alpha\right)\Lambda\left(t\right)\right\}.$$

Thus, for $0 < t_1 \le t_2 \le \cdots \le t_m \le t$,

$$\begin{aligned} f(t_1, t_2, \ldots, t_m \mid n, m) &= \frac{f(t_1, t_2, \ldots, t_m, n, m)}{P(N(t) = n, M(t) = m)} \\ &= m! \prod_{i=1}^{m} \left(\frac{\alpha \lambda(t_i) \exp\{\alpha \Lambda(t_i)\}}{\exp\{\alpha \Lambda(t)\} - 1} \right) \\ &= f(t_1, t_2, \ldots, t_m \mid m). \end{aligned}$$

Obviously, $(T_1, T_2, \ldots, T_{M(t)} \mid N(t), M(t))$ does not depend on $N(t)$.

In the described model, we consider two types of repair attempts: unsuccessful and successful. As mentioned previously, successful repair attempts results in worse-than-minimal repairs of the system and are, therefore, modelled by the GPP. That is, the distributional properties of $M(t)$ are known. Now, we will derive the distribution of the unsuccessful repair attempts, $N(t) - M(t)$, and the corresponding mean. These characteristics play an important role when planning maintenance actions for a system subject to multiple repair attempts.

Proposition 3.4. The distribution and the mean of $N(t) - M(t)$ is given by

$$\begin{aligned} P(N(t) - M(t) = 0) &= \frac{1}{((1-\rho)\exp\{\alpha\Lambda(t)\} + \rho)^{\frac{1}{\alpha}}}, \\ P(N(t) - M(t) = k) &= \sum_{m=1}^{\infty} \binom{m+k-1}{m-1} \rho^m (1-\rho)^k \frac{\Gamma\left(\frac{1}{\alpha} + m\right)}{\Gamma\left(\frac{1}{\alpha}\right) m!} \\ &\quad \times (1 - \exp\{-\alpha\Lambda(t)\})^m (\exp\{-\Lambda(t)\}), \quad \text{for } k = 1, 2, \ldots, \end{aligned}$$

and

$$E[N(t) - M(t)] = \frac{\exp\{\alpha\Lambda(t)\} - 1}{\alpha} \left(\frac{1-\rho}{\rho} \right),$$

respectively.

Proof. Observe that

$$\begin{aligned} P(N(t) - M(t) = 0) &= P(M(t) = 0) + \sum_{m=1}^{\infty} P(N(t) = m \mid M(t) = m) P(M(t) = m) \\ &= (\exp\{-\Lambda(t)\}) + (\exp\{-\Lambda(t)\}) \sum_{m=1}^{\infty} \frac{\Gamma\left(\frac{1}{\alpha} + m\right)}{\Gamma\left(\frac{1}{\alpha}\right) m!} (\rho (1 - \exp\{-\alpha\Lambda(t)\}))^m \\ &= (\exp\{-\Lambda(t)\}) \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{1}{\alpha} + m\right)}{\Gamma\left(\frac{1}{\alpha}\right) m!} (\rho (1 - \exp\{-\alpha\Lambda(t)\}))^m \\ &= \frac{1}{((1-\rho)\exp\{\alpha\Lambda(t)\} + \rho)^{\frac{1}{\alpha}}} \end{aligned}$$

and

$$P\left(N\left(t\right)-M\left(t\right)=k\right)=\sum_{m=1}^{\infty}P\left(N\left(t\right)=m+k\mid M\left(t\right)=m\right)P\left(M\left(t\right)=m\right)$$
$$=\sum_{m=1}^{\infty}\binom{m+k-1}{m-1}\rho^{m}\left(1-\rho\right)^{k}\frac{\Gamma\left(\frac{1}{\alpha}+m\right)}{\Gamma\left(\frac{1}{\alpha}\right)m!}$$
$$\times\left(1-\exp\left\{-\alpha\Lambda\left(t\right)\right\}\right)^{m}\left(\exp\left\{-\Lambda\left(t\right)\right\}\right),\qquad\text{for } k=1,2,\ldots.$$

Also, it is obvious that

$$E\left[N\left(t\right)-M\left(t\right)\right]=E\left[N\left(t\right)\right]-E\left[M\left(t\right)\right]=\frac{\exp\left\{\alpha\Lambda\left(t\right)\right\}-1}{\alpha}\left(\frac{1-\rho}{\rho}\right).$$ ■

4. Further generalisation

In this section, we consider a further generalisation to the process described in Section 3. Suppose now that each attempt to repair the system from a failure at time t is unsuccessful with probability $(1-\rho(t))$ and is successful with probability $\rho(t)$, independently. That is, consider the case where the probability of a successful repair now depends on time. This is an important practical extension to the previously defined model and can describe various real-life situations. For example, as an item ages, it may become more difficult to execute a successful repair action. This could be incorporated through a decreasing with time probability $\rho(t)$. Unlike the time-independent case, deriving the general properties of this process in closed form is not possible. However, we can obtain the corresponding means for the processes $N(t)$, $M(t)$, and $L(t)\equiv N(t)-M(t)$ using the intensity function of a counting process. Recall that the intensity function of a counting process $\{Y(t),t\geq 0\}$ is defined by

$$\varphi_{Y}\left(t\right)=\frac{\mathrm{d}E\left[Y\left(t\right)\right]}{\mathrm{d}t}.$$

That is, if we can derive $\varphi_Y(t)$, $E[Y(t)]$ can easily be obtained as

$$E\left[Y\left(t\right)\right]=\int_{0}^{t}\varphi_{Y}\left(u\right)\mathrm{d}u.$$

The intensity functions $\varphi_N(t)$, $\varphi_M(t)$, and $\varphi_L(t)$ for the processes $N(t)$, $M(t)$ and $L(t)$, respectively, as well as their corresponding means are given in Proposition 4.1.

Proposition 4.1. The intensity functions $\varphi_N(t)$, $\varphi_M(t)$, and $\varphi_L(t)$ are given by

$$\varphi_{N}\left(t\right)=\frac{\lambda\left(t\right)\exp\left\{\alpha\Lambda\left(t\right)\right\}}{\rho\left(t\right)},\qquad\varphi_{M}\left(t\right)=\lambda\left(t\right)\exp\left\{\alpha\Lambda\left(t\right)\right\},$$

and

$$\varphi_{L}\left(t\right)=\lambda\left(t\right)\exp\left\{\alpha\Lambda\left(t\right)\right\}\left(\frac{1-\rho\left(t\right)}{\rho\left(t\right)}\right),$$

with corresponding mean values

$$E\left[N\left(t\right)\right]=\int_{0}^{t}\frac{\lambda\left(u\right)\exp\left\{\alpha\Lambda\left(u\right)\right\}}{\rho\left(u\right)}\mathrm{d}u,\qquad E\left[M\left(t\right)\right]=\int_{0}^{t}\lambda\left(u\right)\exp\left\{\alpha\Lambda\left(u\right)\right\}\mathrm{d}u,$$

and

$$E\left[L\left(t\right)\right] = \int_0^t \lambda\left(u\right)\exp\left\{\alpha\Lambda\left(u\right)\right\}\left(\frac{1-\rho\left(u\right)}{\rho\left(u\right)}\right)\mathrm{d}u.$$

Proof. To simplify notation, define $N(t,t+\Delta t) = N(t+\Delta t) - N(t)$, $M(t,t+\Delta t) = M(t+\Delta t) - M(t)$. Further, let S_i, $i = 1, 2, \dots$ denote the ith occurrence time of the GPP $\{M(t), t \geq 0\}$ in the interval $(t, t+\Delta t]$. Then,

$$N\left(t,t+\Delta t\right) = Y_1 + Y_2 + \cdots + Y_{M(t,t+\Delta t)},$$

where the Y_i, $i = 1, 2, \dots$ are independent geometric random variables with corresponding probability of success $\rho(s_i)$.

Observe that

$$E\left[N\left(t,t+\Delta t\right)\right] = E\left[E\left[N\left(t,t+\Delta t\right) \mid M\left(t,t+\Delta t\right)\right]\right]$$

and

$$\begin{aligned}&E\left[N\left(t,t+\Delta t\right) \mid M\left(t,t+\Delta t\right)\right]\\ =&E_{S_1,S_2,\dots,S_{M(t,t+\Delta t)}\mid M(t,t+\Delta t)}\left[E\left[N\left(t,t+\Delta t\right) \mid S_1,S_2,\dots,S_{M(t,t+\Delta t)},M\left(t,t+\Delta t\right)\right]\right]\end{aligned}$$

where $E_{S_1,S_2,\dots,S_{M(t,t+\Delta t)}\mid M(t,t+\Delta t)}[\cdot]$ represents the expectation with respect to the conditional joint distribution of $(S_1, S_2, \dots, S_{M(t,t+\Delta t)} \mid M(t, t+\Delta t))$.

Now,

$$\begin{aligned}&E\left[N\left(t,t+\Delta t\right) \mid S_1 = s_1, S_2 = s_2, \dots, S_{M(t,t+\Delta t)} = s_m, M\left(t,t+\Delta t\right) = m\right]\\ =&E\left[Y_1\left(s_1\right) + Y_2\left(s_2\right) + \cdots + Y_m\left(s_m\right)\right],\end{aligned}$$

where $Y_i(s_i)$ is the geometric random variable with corresponding success probability $\rho(s_i)$. Thus,

$$E\left[N\left(t,t+\Delta t\right) \mid S_1 = s_1, S_2 = s_2, \dots, S_{M(t,t+\Delta t)} = s_m, M\left(t,t+\Delta t\right) = m\right] = \sum_{i=1}^{m}\frac{1}{\rho\left(s_i\right)}$$

and

$$E\left[N\left(t,t+\Delta t\right) \mid M\left(t,t+\Delta t\right)\right] = E_{S_1,S_2,\dots,S_{M(t,t+\Delta t)}\mid M(t,t+\Delta t)}\left[\sum_{i=1}^{M(t,t+\Delta t)}\frac{1}{\rho\left(s_i\right)}\right].$$

By Theorem 3 in Cha (2014), the conditional joint distribution of $(S_1, S_2, \dots, S_m \mid M\left(t,t+\Delta t\right) = m)$ is given by

$$m!\prod_{i=1}^{m}\left(\frac{\alpha\lambda\left(s_i\right)\exp\left\{\alpha\Lambda\left(s_i\right)\right\}}{\exp\left\{\alpha\Lambda\left(t+\Delta t\right)\right\} - \exp\left\{\alpha\Lambda\left(t\right)\right\}}\right), \quad t < s_1 < s_2 < \cdots < s_m < t+\Delta t.$$

Hence, the conditional joint distribution of $(S_1, S_2, \dots, S_m \mid M(t, t+\Delta t) = m)$ is the same as the joint distribution of $V_{(1)}, V_{(2)}, \dots, V_{(m)}$, where $V_{(1)} \leq V_{(2)} \leq \dots \leq V_{(m)}$ are the order statistics of iid random variables $V_1, V_2, \dots, V_m$, which share common pdf

$$\frac{\alpha\lambda\left(v\right)\exp\left\{\alpha\Lambda\left(v\right)\right\}}{\exp\left\{\alpha\Lambda\left(t+\Delta t\right)\right\} - \exp\left\{\alpha\Lambda\left(t\right)\right\}}, \quad t < v \leq t+\Delta t.$$

Thus,

$$\begin{aligned}
E_{S_1,S_2,\ldots,S_{M(t,t+\Delta t)}|M(t,t+\Delta t)}\left[\sum_{i=1}^{M(t,t+\Delta t)}\frac{1}{\rho(s_i)}\right] &= E\left[\sum_{i=1}^{m}\frac{1}{\rho\left(V_{(i)}\right)}\right] \\
&= E\left[\sum_{i=1}^{m}\frac{1}{\rho(V_i)}\right] \\
&= m\int_0^{t+\Delta t}\frac{1}{\rho(v)}\frac{\alpha\lambda(v)\exp\{\alpha\Lambda(v)\}}{\exp\{\alpha\Lambda(t+\Delta t)\}-\exp\{\alpha\Lambda(t)\}}\mathrm{d}v.
\end{aligned}$$

Finally,

$$\begin{aligned}
E\left[N(t,t+\Delta t)\right] &= E\left[M(t,t+\Delta t)\int_0^{t+\Delta t}\frac{1}{\rho(v)}\frac{\alpha\lambda(v)\exp\{\alpha\Lambda(v)\}}{\exp\{\alpha\Lambda(t+\Delta t)\}-\exp\{\alpha\Lambda(t)\}}\mathrm{d}v\right] \\
&= E\left[M(t,t+\Delta t)\right]\int_0^{t+\Delta t}\frac{1}{\rho(v)}\frac{\alpha\lambda(v)\exp\{\alpha\Lambda(v)\}}{\exp\{\alpha\Lambda(t+\Delta t)\}-\exp\{\alpha\Lambda(t)\}}\mathrm{d}v \\
&= \int_0^{t+\Delta t}\frac{\lambda(v)\exp\{\alpha\Lambda(v)\}}{\rho(v)}\mathrm{d}v.
\end{aligned}$$

From the definition of the intensity function,

$$\begin{aligned}
\varphi_N(t) &= \frac{\mathrm{d}E\left[N(t)\right]}{\mathrm{d}t} \\
&= \lim_{\Delta t\to 0}\frac{E\left[N(t,t+\Delta t)\right]}{\Delta t} \\
&= \lim_{\Delta t\to 0}\frac{\int_0^{t+\Delta t}\frac{\lambda(v)\exp\{\alpha\Lambda(v)\}}{\rho(v)}\mathrm{d}v}{\Delta t} \\
&= \frac{\lambda(t)\exp\{\alpha\Lambda(t)\}}{\rho(t)}.
\end{aligned}$$

Obviously, $\varphi_M(t) = \lambda(t)\exp\{\alpha\Lambda(t)\}$, and

$$\varphi_L(t) = \varphi_N(t) - \varphi_M(t) = \lambda(t)\left(\frac{1-\rho(t)}{\rho(t)}\right)\exp\{\alpha\Lambda(t)\}.$$

The corresponding means can be found by integrating the respective intensity functions. ■

Remark 1. We have considered the case where the successful repairs are modelled by the corresponding GPP and are thus considered to be worse-than-minimal repairs. In Cha and Finkelstein (2023), the successful repairs are modelled by the corresponding NHPP and are thus considered to be at least minimal. As the NHPP is a special case of the GPP, the results presented in Proposition 4.1 can be considered as a further generalisation of the results presented in Cha and Finkelstein (2023). To obtain the results of this special case from our model, we take the limit as α tends to 0. This results in intensity functions given by

$$\varphi_N(t) = \frac{\lambda(t)}{\rho(t)},\ \varphi_M(t) = \lambda(t) \quad \text{and} \quad \varphi_L(t) = \lambda(t)\left(\frac{1-\rho(t)}{\rho(t)}\right),$$

as per Cha and Finkelstein (2023).

5. Application to optimal age replacement

Consider a system with increasing baseline failure rate (failure rate before the first failure/successful repair) $\lambda(t)$. This is a common assumption in optimal maintenance problems as it describes the natural degradation or aging of a system with time (see Nakagawa, 2005). On each failure, the repair action is initiated. This repair action is successful with decreasing probability $\rho(t)$, resulting in a worse-than-minimal repair, or it is unsuccessful with probability $1-\rho(t)$, resulting in another repair attempt, which can itself be either successful or unsuccessful, independently, and so on. For simplicity, we will assume that the time from failure until a successful worse-than-minimal repair is negligible. Suppose that a system is replaced with a new one when it reaches age T (periodic replacement policy). That is, after each replacement, a new cycle begins, and so on. Therefore, it is sufficient to consider only the expected cost rate on one first cycle.

Let c_u, c_s, and c_r be the cost of an unsuccessful repair attempt, a successful worse-than-minimal repair, and replacement, respectively. Then, from Proposition 4.1, the long-run expected cost rate, which is often used for the cost-wise characterisation of the preventive maintenance policy, is given by

$$c(T) = \frac{c_u \int_0^T \lambda(u)\left(\frac{1-\rho(u)}{\rho(u)}\right)\exp\{\alpha\Lambda(u)\}\,\mathrm{d}u + c_s\int_0^T \lambda(u)\exp\{\alpha\Lambda(u)\}\,\mathrm{d}u + c_r}{T}.$$

If a system does not age, then there is no need to replace it. Therefore, the increasing baseline failure rate and decreasing probability of successful worse-than-minimal repair assumptions guarantee the existence of an optimal replacement time. The existence of the optimal replacement time, under these assumptions, is given in Proposition 5.1.

Proposition 5.1. If

(i) $\rho(t)$ is strictly decreasing to 0 and $\lambda(t)$ is strictly increasing, or

(ii) $\rho(t)$ is strictly decreasing and $\lambda(t)$ is strictly increasing to ∞,

then there is a unique solution T^* which minimises $c(T)$.

Proof. For simplicity, let

$$\Psi(T) = T\left\{c_u\lambda(T)\left(\frac{1-\rho(T)}{\rho(T)}\right)\exp\{\alpha\Lambda(T)\} + c_s\lambda(T)\exp\{\alpha\Lambda(T)\}\right\}$$
$$-\left\{c_u\int_0^T \lambda(u)\left(\frac{1-\rho(u)}{\rho(u)}\right)\exp\{\alpha\Lambda(u)\}\,\mathrm{d}u + c_s\int_0^T \lambda(u)\exp\{\alpha\Lambda(u)\}\,\mathrm{d}u + c_r\right\}.$$

Then,

$$c'(T) = \frac{1}{T^2}\Psi(T).$$

It can easily be seen that

$$\Psi(0) = -c_r < 0.$$

Further, under the given assumptions,

$$\Psi'(T) = T\Bigg(c_u\left\{\frac{\mathrm{d}}{\mathrm{d}T}\left[\lambda(T)\left(\frac{1-\rho(T)}{\rho(T)}\right)\right]\exp\{\alpha\Lambda(T)\} + \alpha\lambda(T)^2\left(\frac{1-\rho(T)}{\rho(T)}\right)\exp\{\alpha\Lambda(T)\}\right\} + c_s\left\{\lambda'(T)\exp\{\alpha\Lambda(T)\} + \alpha\lambda(T)^2\exp\{\alpha\Lambda(T)\}\right\}\Bigg) > 0,$$

and thus $\Psi(T)$ is increasing for all $T > 0$.

To aid in the remainder of this proof, $\Psi(T)$ can be rewritten as

$$\Psi(T) = c_u\int_0^\infty \left\{\lambda(T)\left(\frac{1-\rho(T)}{\rho(T)}\right)\exp\{\alpha\Lambda(T)\} - \lambda(u)\left(\frac{1-\rho(u)}{\rho(u)}\right)\exp\{\alpha\Lambda(u)\}\right\}\mathbf{1}_u[0,T]\,\mathrm{d}u + c_s\int_0^\infty \{\lambda(T)\exp\{\alpha\Lambda(T)\} - \lambda(u)\exp\{\alpha\Lambda(u)\}\}\mathbf{1}_u[0,T]\,\mathrm{d}u - c_r$$

by rearranging the terms. Then, under the given assumptions,

$$\lim_{T\to\infty}\lambda(T)\left(\frac{1-\rho(T)}{\rho(T)}\right)\exp\{\alpha\Lambda(T)\} = \infty,$$

and

$$\int_0^\infty \{\lambda(T)\exp\{\alpha\Lambda(T)\} - \lambda(u)\exp\{\alpha\Lambda(u)\}\}\mathbf{1}_u[0,T]\,\mathrm{d}u > 0,\ \forall T > 0.$$

Thus, by the monotone convergence theorem,

$$\lim_{T\to\infty}\Psi(T) = \infty.$$

Therefore, as $\Psi(T)$ is increasing from $-c_r$ to ∞, a unique solution T^* which solves the equation $c'(T) = 0$ exists. ■

Numerical illustrations

For convenience, we will consider the case where $c_u = c_s$ to demonstrate the optimal replacement results. Note that in this case the objective function can be rewritten as

$$\frac{c(T)}{c_u} = \frac{\int_0^T \lambda(u)\left(\frac{1-\rho(u)}{\rho(u)}\right)\exp\{\alpha\Lambda(u)\}\,\mathrm{d}u + \int_0^T \lambda(u)\exp\{\alpha\Lambda(u)\}\,\mathrm{d}u + \frac{c_r}{c_u}}{T}.$$

First, let us consider the case where $\rho(t) \equiv 1$. This can be considered as the classical case without multiple attempts. Let $\lambda(t) = 0.1t$, $c_r = 10$, $c_u = 1, 2, 3$, and $\alpha = 0, 0.1, 0.2, 0.5$. Note here that the NHPP ($\alpha = 0$) curve is equivalent to the multi-attempt minimal repair case discussed in Cha and Finkelstein (2023). It is clear from Figure 1 and all subsequent figures, that an optimal replacement time T^* exists. Further, the optimal replacement time T^* is decreasing for increasing α. This conforms with the general reasoning. Previously conducted repairs having more influence on the system's susceptibility to failure would suggest replacing the system sooner. Similarly, the optimal replacement time T^* is decreasing for increasing c_u. The higher cost of a repair would mean that we should replace the system more often.

Table 1. T^* for $\rho(t) \equiv 1$, $\lambda(t) = 0.1t$, $c_r = 10$, $c_u = 1,2,3$, and $\alpha = 0, 0.1, 0.2, 0.5$

$c_u \backslash \alpha$	0	0.1	0.2	0.5
1	14.14	10.00	8.60	6.75
2	10.00	7.98	7.07	5.75
3	8.16	6.88	6.22	5.18

Table 2. T^* for $\rho(t) = \exp\{-t/5\}$, $\lambda(t) = 0.1t$, $c_r = 10$, $c_u = 1,2,3$, and $\alpha = 0, 0.1, 0.2, 0.5$

$c_u \backslash \alpha$	0	0.1	0.2	0.5
1	6.45	6.05	5.74	5.11
2	5.24	5.00	4.79	4.35
3	4.61	4.43	4.28	3.93

Table 3. T^* for $\rho(t) = \exp\{-t/5\}$, $\lambda(t) = 0.1(t^2 + 1)$, $c_r = 10$, $c_u = 1,2,3$, and $\alpha = 0, 0.1, 0.2, 0.5$

$c_u \backslash \alpha$	0	0.1	0.2	0.5
1	3.94	3.70	3.52	3.17
2	3.27	3.13	3.02	2.78
3	2.93	2.83	2.74	2.55

Table 4. T^* for $\rho(t) = \exp\{-t/2\}$, $\lambda(t) = 0.1t$, $c_r = 10$, $c_u = 1,2,3$, and $\alpha = 0, 0.1, 0.2, 0.5$

$c_u \backslash \alpha$	0	0.1	0.2	0.5
1	4.16	4.07	3.99	3.77
2	3.51	3.45	3.40	3.25
3	3.16	3.11	3.07	2.95

The optimal replacement times T^* for the various combinations of α and c_u in the classical case are given in Table 1. These results also support that, for increasing α and increasing c_u, T^* is decreasing.

In Figure 2 and Table 2 the probability of a successful repair attempt is changed to $\rho(t) = \exp\{-t/5\}$. We see that the times for optimal replacement T^* under all combinations of c_u and α are smaller than those in the classical case with $\rho(t) \equiv 1$. This is intuitively clear as the decreasing probability of a successful repair introduces certain 'aging' into the system, and as such, we should replace the system sooner.

In Figure 3 and Table 3 the baseline failure rate is now changed to $\lambda(t) = 0.1(t^2 + 1)$. We see that the times for optimal replacement T^* under all combinations of c_u and α are once again smaller than those in Figure 2 and Table 2, and therefore are smaller than those in the classical case with $\rho(t) \equiv 1$. Increasing the baseline failure rate introduces additional aging to the system, and as such, replacement should be carried out at an earlier time.

Finally, in Figure 4 and Table 4, the baseline failure rate is once again given by $\lambda(t) = 0.1t$, and the probability of a successful repair attempt is $\rho(t) = \exp\{-t/2\}$. Comparing the optimal replacement times T^* under the various combinations of c_u and α with those in Figure 2 and Table 2, we can see, as expected, the steeper decrease of $\rho(t)$ results in smaller values of T^* for all cases. Once again, the case where $\rho(t) = \exp\{-t/2\}$ implies more aging to the system, and as such, replacement should occur sooner.

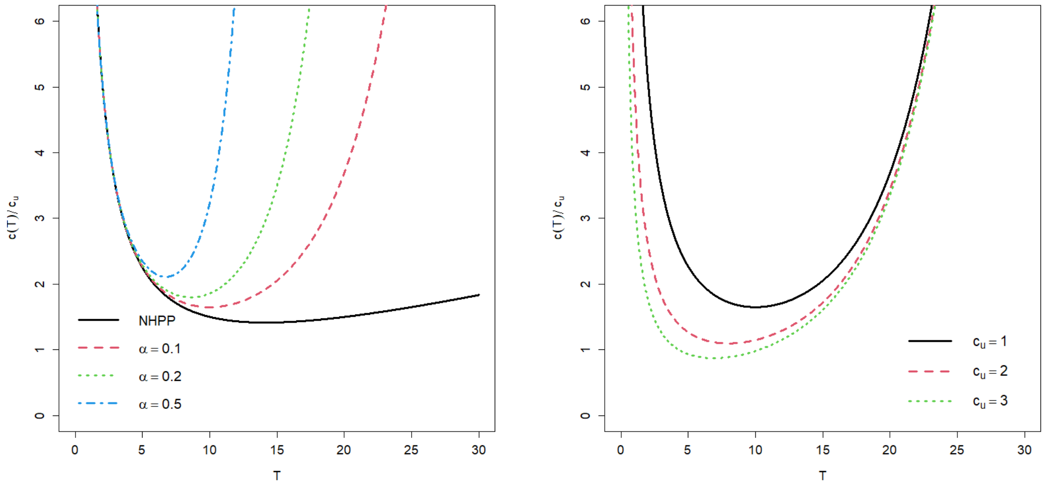

Figure 1. $c(T)/c_u$ for $\rho(t) \equiv 1$, $\lambda(t) = 0.1t$, $c_r = 10$, and varying α with $c_u = 1$ (left), and varying c_u with $\alpha = 0.1$ (right).

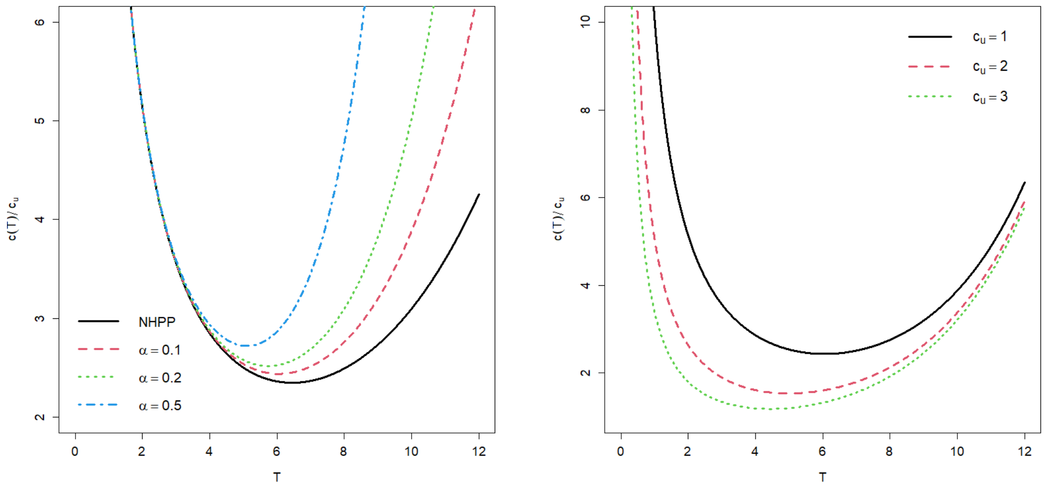

Figure 2. $c(T)/c_u$ for $\rho(t) = \exp\{-t/5\}$, $\lambda(t) = 0.1t$, $c_r = 10$, and varying α with $c_u = 1$ (left), and varying c_u with $\alpha = 0.1$ (right).

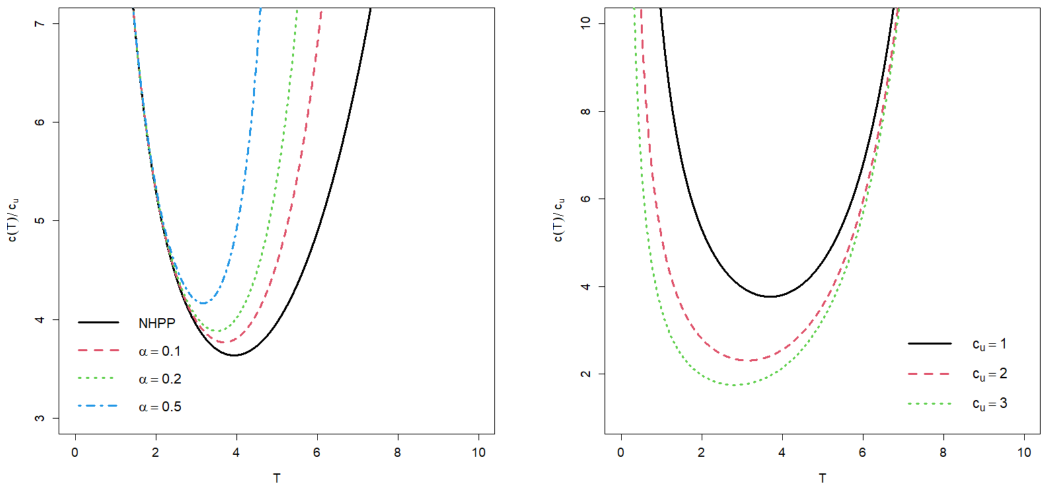

Figure 3. $c(T)/c_u$ for $\rho(t) = \exp\{-t/5\}$, $\lambda(t) = 0.1(t^2 + 1)$, $c_r = 10$, and varying α with $c_u = 1$ (left), and varying c_u with $\alpha = 0.1$ (right).

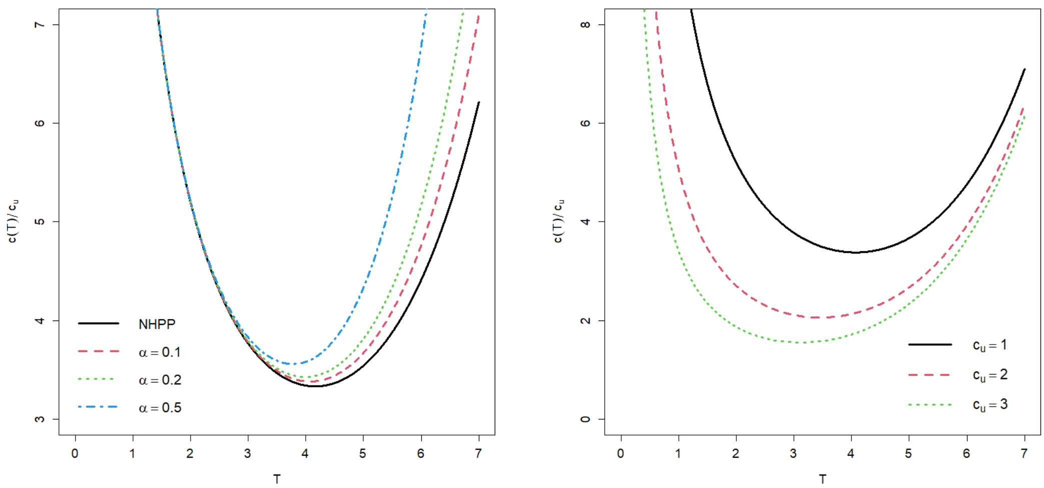

Figure 4. $c(T)/c_u$ for $\rho(t) = \exp\{-t/2\}$, $\lambda(t) = 0.1t$, $c_r = 10$, and varying α with $c_u = 1$ (left), and varying c_u with $\alpha = 0.1$ (right).

6. Conclusions

In practice, an attempt to repair a failed system can be unsuccessful and, therefore, will need to be repeated until a successful repair has been made. These unsuccessful repair attempts can occur due to a number of reasons, such as faulty spare parts, system design flaws, the failure of another system component while replacing the failed one, errors in fault detection, and so on. Therefore, to model maintenance of various real-world systems, unsuccessful repair attempts are an important practical consideration. Beyond taking into account the possibility of multiple repair attempts, it is also important to consider that the successful repairs to the system may, in fact, be worse-than-minimal. This could occur due to the adverse effects of the previous repair attempts, external and internal shocks to the system, insufficient quality of a successful repair attempt, and so on. Therefore, in this paper, a general multiple-repair-attempt process was defined to account for situations where this may occur.

The multiple-repair-attempt process was derived under the assumption that the underlying process for successful repair attempts is the GPP. This process is becoming well-known in the literature to model repairs that are considered to be worse-than-minimal. Under this assumption, we derive a number of analytical properties that allow for the consideration of the corresponding age replacement problem. The existence of an optimal replacement time is also discussed, and detailed numerical examples under various settings are provided. We also have shown that the multi-attempt minimal repair model, discussed in Cha and Finkelstein (2023), is a special case to our more general model.

Further generalisation of this multiple-repair-attempt model to the case of imperfect repair could become technically challenging without the use of additional assumptions and simplifications. However, we believe there is still potential for further research in relation to the preventative maintenance policy under this model. For example, the policy could be generalised to consider the case where replacement is carried out when the system reaches age T or the number of unsuccessful attempts during one repair action reach a predetermined threshold, say k, whichever comes first. Alternatively, replacement is carried out when the system reaches age T or the cumulative number of unsuccessful repair attempts reaches a predetermined threshold, say c, whichever comes first.

References

AVEN, T. AND JENSEN, U. (1999). *Stochastic Models in Reliability*. Springer, New York, NY.

AVEN, T. AND JENSEN, U. (2000). A general minimal repair model. *Journal of Applied Probability*, **37**, 187–197.

BADÍA, F. G., BERRADE, M. D., CHA, J. H., AND LEE, H. (2018). Optimal replacement policy under a general failure and repair model: Minimal versus worse than old repair. *Reliability Engineering & System Safety*, **180**, 362–372.

BADÍA, F. G., BERRADE, M. D., AND LEE, H. (2020). An study of cost effective maintenance policies: Age replacement versus replacement after N minimal repairs. *Reliability Engineering & System Safety*, **201**, 106949.

BARLOW, R. AND HUNTER, L. (1960). Optimum preventive maintenance policies. *Operations Research*, **8**, 90–100.

BARLOW, R. E. AND PROSCHAN, F. (1975). *Statistical Theory of Reliability and Life Testing: Proba-*

bility Models. Holt, Rinehart and Winston, New York, NY.

Beichelt, F. and Fischer, K. (1980). General failure model applied to preventive maintenance policies. *IEEE Transactions on Reliability*, **29**, 39–41.

Cha, J. H. (2014). Characterization of the generalized Polya process and its applications. *Advances in Applied Probability*, **46**, 1148–1171.

Cha, J. H. and Finkelstein, M. (2018). *Point Processes for Reliability Analysis: Shocks and Repairable Systems*. Springer, Switzerland.

Cha, J. H. and Finkelstein, M. (2023). On the multi-attempt minimal repair and the corresponding counting process. *Applied Stochastic Models in Business and Industry*, **40**, 206–215.

Cha, J. H. and Finkelstein, M. (2024). Preventive maintenance for the constrained multi-attempt minimal repair. *Reliability Engineering & System Safety*, **243**, 109899.

Cha, J. H., Finkelstein, M., and Levitin, G. (2023). On the delayed worse-than-minimal repair model and its application to preventive replacement. *IMA Journal of Management Mathematics*, **34**, 101–122.

Chukova, S. and Minkova, L. D. (2019). Non-homogeneous Pólya-Aeppli process. *Communications in Statistics - Simulation and Computation*, **48**, 2955–2967.

Finkelstein, M. (2008). *Failure Rate Modelling for Reliability and Risk*. Springer.

Finkelstein, M. and Cha, J. H. (2013). *Stochastic Modeling for Reliability: Shocks, Burn-in and Heterogeneous Populations*. Springer, London.

Lee, H. and Cha, J. H. (2016). New stochastic models for preventive maintenance and maintenance optimization. *European Journal of Operational Research*, **255**, 80–90.

Liu, P. and Wang, G. (2021). Optimal periodic preventive maintenance policies for systems subject to shocks. *Applied Mathematical Modelling*, **93**, 101–114.

Minkova, L. D. (2004). The Pólya-Aeppli process and ruin problems. *Journal of Applied Mathematics and Stochastic Analysis*, **2004**, 221–234.

Nakagawa, T. (1980). A summary of imperfect preventive maintenance policies with minimal repair. *RAIRO - Operations Research*, **14**, 249–255.

Nakagawa, T. (1986). Periodic and sequential preventive maintenance policies. *Journal of Applied Probability*, **23**, 536–542.

Nakagawa, T. (2005). *Maintenance Theory of Reliability*. Springer, London.

Park, D. H., Jung, G. M., and Yum, J. K. (2000). Cost minimization for periodic maintenance policy of a system subject to slow degradation. *Reliability Engineering & System Safety*, **68**, 105–112.

Tadj, L., Ouali, M.-S., Yacout, S., Ait-Kadi, D., et al. (2011). *Replacement Models with Minimal Repair*. Springer, London.

Manuscript received 2023-08-08, revised 2024-02-08, accepted 2024-03-07.