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Testing for no effect in the spatial functional linear regression model

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We consider a functional linear regression model with a real-valued response and a functional random variable with its derivative as covariates. We are interested in testing the null hypothesis of no covariates effect using a spatially dependent sample. We propose two test statistics which take into account the proximity between sites and we establish the asymptotic normality of cross covariance operator between both interest variables. From this result, we derive asymptotic distributions of these both statistics. Then, we illustrate our test procedure by means of a simulation study.

Keywords: Functional data, Spatial data, Spatial functional linear regression, Testing hypotheses.

1. Introduction

Since the last decade, big data with dynamic components in time and space are becoming more and more popular in a number of disciplines such as environmental sciences, neuroimaging and genomics, epidemiology, and hydrology. Functional data analysis (FDA) is well suited to deal with such data and the linear regression model remains a fundamental tool for modelling relationships between a real-valued response variable and an explanatory variable which is of functional nature. Since the seminal paper of Hastie and Mallows (1993) on functional linear models with a scalar response, several types of functional linear models for independent or time-series data have been developed for different purposes (see, e.g. Cardot et al., 2003; Liu et al., 2017; Mas and Pumo, 2009; Aue et al., 2015). Compared to independent data, functional linear regression with spatially dependent high-dimensional data is still an open research problem (see, e.g., Menafoglio and Secchi (2017), Giraldo et al. (2018)). Functional spatial linear prediction using kriging methods have been tackled by Bohorquez et al. (2017), Giraldo et al. (2018). While the one based on spatial autoregressive functional models have been considered by Ruiz-Medina (2011, 2012). An extension of the functional linear regression model with a functional variable and its first derivative as covariates, studied in Mas and Pumo (2009), to the case of spatially dependent data has been introduced in Bouka et al. (2018). Such models may be used in many applications; for instance in environmental domain for

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forecasting of ozone pollution (see Bouka et al. (2023)). In this paper, we consider the model, as in Bouka et al. (2018), defined by

$$Y_{\mathbf{i}} = \langle \beta, X_{\mathbf{i}} \rangle_H + \left\langle \gamma, X'_{\mathbf{i}} \right\rangle + \varepsilon_{\mathbf{i}}, \quad \mathbf{i} \in E \subset \mathbb{Z}^d, \quad d \geq 2, \tag{1}$$

where $\{Y_{\mathbf{i}} \in \mathbb{R},\ \mathbf{i} \in E \subset \mathbb{Z}^d\}$ is a real-valued spatial process, $\{X_{\mathbf{i}}(.),\ \mathbf{i} \in E \subset \mathbb{Z}^d\}$ is a spatial functional process with its first derivative $\{X'_{\mathbf{i}}(.),\ \mathbf{i} \in E \subset \mathbb{Z}^d\}$, $\{\varepsilon_{\mathbf{i}},\ \mathbf{i} \in E \subset \mathbb{Z}^d\}$ is a noise independent of $\{X_{\mathbf{i}}(.),\ \mathbf{i} \in E \subset \mathbb{Z}^d\}$ and $\{X'_{\mathbf{i}}(.),\ \mathbf{i} \in E \subset \mathbb{Z}^d\}$. All the random variables of the previous processes are assumed centered, the bivariate process $\{(X_{\mathbf{i}}, Y_{\mathbf{i}})\}_{\mathbf{i} \in E \subset \mathbb{Z}^d}$ is strictly stationary and defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, valued into $H \times \mathbb{R}$ with the same distribution as a random vector $(X,\ Y)$. Here H is the Sobolev space, defined by $H = \{x \in G, x' \in G\}$ where $G := L^2[0, 1]$ is the Hilbert space of squared integrable functions, x' stands for the first derivative of $x \in H$. Let $\langle ., . \rangle_H$ and $\langle ., . \rangle$ be the inner products in H and G respectively defined by

$$\langle f, g \rangle_H = \int_0^1 f(t)g(t)dt + \int_0^1 f'(t)g'(t)dt, \quad \langle f, g \rangle = \int_0^1 f(t)g(t)dt;$$

the associated norms are denoted by $\|.\|_H$ and $\|.\|_G$ respectively. We are interested in testing for the nullity of the pair (β, γ) in (1). That is testing whether there is not a linear connection between the functional variables $(X_{\mathbf{i}}, X'_{\mathbf{i}})$ and the real variable $Y_{\mathbf{i}}$ according to relation (1). For that, we consider the following test:

$$H_0 : \beta = 0 \text{ and } \gamma = 0 \quad \text{vs.} \quad H_1 :\ \beta \neq 0 \text{ or } \gamma \neq 0. \tag{2}$$

Despite the potential applications of such no regression effect for spatially dependent data, this problem has not been addressed in the literature to the best of our knowledge. For the basic functional linear model $Y = \langle \beta, X \rangle + \varepsilon$, Cardot et al. (2003) proposed to test the hypothesis $H_0 : \beta = 0$, against $H_1 :\ \beta \neq 0$ for independent data.

Cardot et al. (2004) extended the previous work by considering an additive functional linear regression model and a test for partial nullity of the involved coefficients. Hilgert et al. (2013) considered the basic functional linear model and proposed a test statistic that does not depend on the number of principal components compared to Cardot et al. (2003). Kong et al. (2016) considered the case where the functional covariate is contaminated by measurement error and may be observed densely.

In this paper, we first extend the work of Cardot et al. (2003) to the spatial context and then add an effect of the first derivative of the functional covariate in the model. The rest of the paper is organised as follows. Section 2 is devoted to the proposed tests for no effect in the model (2). The test statistics are defined and asymptotic results are given under some mild conditions. Section 3 presents a simulation study. The conclusion is given in Section 4. The proofs of the main results are postponed to Section 5.

2. Testing for no effect

In this section, we consider the spatial functional linear regression model with derivatives defined in (1). As mentioned previously, we aim to test the hypothesis H_0 against H_1 as given in (2). For that, let us consider the covariance and cross-covariance operators $\Gamma = \mathbb{E}[X \otimes_H X]$ (where $\otimes_H$ stands for the tensor product defined by $(u \otimes_H v)h =< u, h >_H v)$ and Δ such that $\Delta x = \mathbb{E}(Y\langle X, x\rangle_H)$. We denote by $(\lambda_j)_{j\geq 1}$ the sequence of eigenvalues of Γ and $(v_j)_{j\geq 1}$ the sequence of associated eigenfunctions. We assume that λ_j, $j = 1, 2, \ldots$ are strictly positive and verify $\sum_{j=1}^{+\infty} \lambda_j < +\infty$. Recall that $(v_j)_{j\geq 1}$ is an orthonormal basis of Hilbert space H. Let $\langle ., .\rangle_{\mathcal{H}S}$ be the inner product of the space $\mathcal{H}S$ of Hilbert-Schmidt operators, defined by $\langle T, S\rangle_{\mathcal{H}S} = \sum_{j=1}^{+\infty}\langle Tv_j, Sv_j\rangle$ and let $\|.\|_{\mathcal{H}S}$ be its associated norm. Before giving the test statistic, let us state a relationship between the operators Δ and Γ. This relationship is obtained by a direct calculation.

Proposition 1. *Let Ψ be the continuous linear operator from H to $\mathbb{R}$ defined by $\Psi(.) = \langle\beta, .\rangle_H + \langle D^*\gamma, .\rangle$ where D^* stands for the adjoint of the differential operator D. Then we have $\Delta = \Psi\Gamma$.*

Under H_0, we have $\Delta = \Psi\Gamma = 0$, that is $\Gamma\Psi^* = 0$, where Ψ^* is the adjoint operator of Ψ. Since the eigenvalues of Γ are strictly positive, we deduce that $\Psi^* = 0$. Since $Y_{\mathbf{i}} = \Psi(X_{\mathbf{i}}) + \varepsilon_{\mathbf{i}}$, testing for $\Psi = 0$ is equivalent to testing for $\Delta = 0$. Let us first study the particular case of functional linear model without derivative; model (1) without the term $\langle\beta', X'_{\mathbf{i}}\rangle + \langle\gamma, X'_{\mathbf{i}}\rangle$, that is

$$Y_{\mathbf{i}} = \langle\beta, X_{\mathbf{i}}\rangle + \varepsilon_{\mathbf{i}}. \tag{3}$$

2.1 Case of the regression model without the derivative

We investigate here the particular test for the hypotheses

$$H_0 : \beta = 0 \quad \text{against} \quad H_1 : \beta \neq 0,$$

corresponding to model (3). We then consider the same problem as in Cardot et al. (2003). The difference comes from the fact that the used sample is spatially dependent compared to the i.i.d case considered in Cardot et al. (2003). As in Cardot et al. (2003), let us decompose the space G as $G = N \oplus W$, with $N = Ker(\Gamma) = \{x \in G : \Gamma x = 0\}$ and $W = \overline{Im(\Gamma)}$ where $Im(\Gamma) = \{z \in G, z = \Gamma x, x \in G\}$ and $\overline{Im(\Gamma)}$ is the closure in G of $Im(\Gamma)$. Since $Y_{\mathbf{i}} = \Psi(X_{\mathbf{i}}) + \varepsilon_{\mathbf{i}}$, the property $\Psi = \Psi_1 + \Psi_2 = 0$ with $(\Psi_1, \Psi_2) \in N \times W$ is equivalent to $\Psi_2 = 0$. Then testing for $\beta = 0$ is equivalent to test for $\Delta = 0$. For giving the proposed test statistics, we consider a spatial sample $(X_{\mathbf{i}}, Y_{\mathbf{i}})_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}$ with

$$\mathcal{I}_{\mathbf{n}} = \{1, 2, \cdots, n_1\} \times \cdots \times \{1, 2, \cdots, n_d\},$$

where $\mathbf{n} = (n_1, \cdots, n_d) \in (\mathbb{N}^*)^d$, $\widehat{\mathbf{n}} = n_1 \times \cdots \times n_d$, $\mathbf{n} \to +\infty$ means that $\min_{i=1,\cdots,d}\{n_i\} \longrightarrow +\infty$ and $|n_i/n_j| < C$ for a constant C such that $0 < C < \infty$ for all i, j such that $1 \leq i, j \leq d$. We consider the empirical operators $\Gamma_{\mathbf{n}}$ and $\Delta_{\mathbf{n}}$ defined by:

$$\Gamma_{\mathbf{n}} x(t) = \frac{1}{\widehat{\mathbf{n}}}\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}\langle X_{\mathbf{i}}, x\rangle X_{\mathbf{i}}(t), \; x \in G, \; t \in [0, 1],$$

$$\Delta_{\mathbf{n}} x = \frac{1}{\widehat{\mathbf{n}}}\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} Y_{\mathbf{i}}\langle X_{\mathbf{i}}, x\rangle, \;\; x \in G.$$

Let $(\widehat{\lambda}_j)_{j\geq 1}$ (resp. $(\widehat{v}_j)_{j\geq 1}$) be the sequence of eigenvalues of $\Gamma_{\mathbf{n}}$ (resp. associated eigenfunctions). We assume that the process $\{Z_{\mathbf{i}}(.) = \langle X_{\mathbf{i}}, .\rangle\varepsilon_{\mathbf{i}}, \; \mathbf{i} \in \mathbb{Z}^d\}$ is spatially mixing (see Tran (1990)). That is

$$\begin{aligned}\alpha(\sigma(Z_{\mathbf{i}}; \mathbf{i} \in K_1), \sigma(Z_{\mathbf{j}}; \mathbf{i} \in K_2)) &= \sup_{A\in\sigma(Z_{\mathbf{i}};\mathbf{i}\in K_1), B\in\sigma(Z_{\mathbf{i}};\mathbf{i}\in K_2)}\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|\} \\ &\leq f(Card(K_1), Card(K_2))\alpha_{1,\infty}(\rho(K_1, K_2)),\end{aligned} \tag{4}$$

where $Card(K_1)$ denotes the cardinality of K_1, ρ stands for the distance defined, for any subsets K_1 and K_2 of $\mathbb{R}^d$, by $\rho(K_1, K_2) = \min\{\|\mathbf{i}-\mathbf{j}\|, \mathbf{i} \in K_1, \mathbf{j} \in K_2\}$ with $\|\mathbf{i}-\mathbf{j}\| = \max_{1\leq s\leq d} |i_s - j_s|$ for any $\mathbf{i}$ and $\mathbf{j}$ in $\mathbb{Z}^d$. In addition, we assume the polynomial mixing condition

$$\alpha_{1,\infty}(u) = O(u^{-\xi}), \;\; \xi > 4d,$$

and that f satisfies either

$$f(n,m) \leq \min\{n,m\} \quad \text{or} \quad f(n,m) \leq C(n+m+1)^{\widetilde{k}},$$

for some $\widetilde{k} > 1$ and some $C > 0$. The results given in the following may be easily extended to exponential rate (i.e $\alpha_{1,\infty}(u) = O(e^{-\xi u})$). Note that if the processes $\{X_{\mathbf{i}}(.), \mathbf{i} \in \mathbb{Z}^d\}$ and $\{\varepsilon_{\mathbf{i}}, \mathbf{i} \in \mathbb{Z}^d\}$ are α-mixing, then it is also the case for $\{Z_{\mathbf{i}}, \mathbf{i} \in \mathbb{Z}^d\}$. Many spatial stochastic processes including time series satisfy strong mixing properties, for more details see for instance Guyon (1995).

We consider the test statistics given by

$$D_{\mathbf{n}} = \frac{1}{\widehat{\sigma}^2} \left\| \widehat{\mathbf{n}}^{1/2} \Delta_{\mathbf{n}} \widehat{B}_{\mathbf{n}} \right\|_{\mathcal{HS}}^2, \tag{5}$$

$$S_{\mathbf{n}} = \frac{1}{\sqrt{q_{\mathbf{n}}}} \left(\frac{1}{\widehat{\sigma}^2} \left\| \widehat{\mathbf{n}}^{1/2} \Delta_{\mathbf{n}} \widehat{B}_{\mathbf{n}} \right\|_{\mathcal{HS}}^2 - q_{\mathbf{n}} \right), \tag{6}$$

where

$$\widehat{B}_{\mathbf{n}}(.) = \sum_{j=1}^{q_{\mathbf{n}}} \widehat{\lambda}_j^{-1/2} \left(\frac{1}{\widehat{\widehat{\mathbf{n}}}} \sum_{\mathbf{i},\boldsymbol{\ell} \in \mathcal{I}_{\mathbf{n}}} h_j(\|\mathbf{i}-\boldsymbol{\ell}\|) \varphi(\|\mathbf{i}-\boldsymbol{\ell}\|) \right)^{-1/2} \left\langle \widehat{v}_j, \cdot \right\rangle \widehat{v}_j,$$

$q_{\mathbf{n}} < \widehat{\mathbf{n}}$, $q_{\mathbf{n}} \longrightarrow +\infty$ as $\mathbf{n} \longrightarrow +\infty$, functions h_j and φ are defined in Assumptions 1 2 below, and $\widehat{\sigma}$ is a consistent estimator of σ such that $\widehat{\mathbf{n}}^{1/2}(\widehat{\sigma}^2 - \sigma^2)$ is bounded in probability. Such an estimator exists; an example is given in Section 3. Note that in the independent sample case with $d = 1$, $D_{\mathbf{n}}$ and $S_{\mathbf{n}}$ become both test statistics studied in Cardot et al. (2003) with $\widehat{B}_n(.) = \sum_{j=1}^{q_n} \widehat{\lambda}_j^{-1/2} \left\langle \widehat{v}_j, \cdot \right\rangle \widehat{v}_j$ and $\delta_1 = 1$. From Theorem 1, When $\mathbf{n}$ is large enough, the χ^2 distribution with $q_{\mathbf{n}}$ degrees of freedom is approximatively the one of $D_{\mathbf{n}}$.

2.2 Assumptions

For establishing asymptotic results on both test statistics, we impose dependency conditions, in particular the mixing and the decay of the cross covariances of the involved processes.

Assumption 1. The random functional process $\{X_{\mathbf{i}}(.), \mathbf{i} \in \mathbb{Z}^d\}$ is strictly stationary, $\|X_{\mathbf{i}}\|_G < M$ a.s with $M > 0$ and for all $\mathbf{i}$, $\boldsymbol{\ell} \in \mathbb{Z}^d$,

$$\mathbb{E}\left(\langle X_{\mathbf{i}}, v_j\rangle \langle X_{\boldsymbol{\ell}}, v_k\rangle\right) = \begin{cases} \lambda_j h_j(\|\mathbf{i}-\boldsymbol{\ell}\|) & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where $h_j : [0, +\infty[\longrightarrow [0, +\infty[$ with $h_j(x) \downarrow 0$, as $x \longrightarrow +\infty$, $h_j(0) = 1$, $\sum_{t=1}^{+\infty} t^{d-1} \sum_{j=1}^{\infty} \lambda_j h_j(t) < \infty$ and $\sum_{j=1}^{+\infty} \lambda_j^{1/4} < +\infty$.

Assumption 2. $(\varepsilon_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ is a stationary isotropic zero-mean random field such that

$$\begin{aligned} Var(\varepsilon_{\mathbf{i}}) &= \sigma^2,\ Cov(\varepsilon_{\mathbf{i}},\varepsilon_{\mathbf{j}}) = \sigma^2\varphi(\|\mathbf{i}-\mathbf{j}\|) \ \text{ for } \ \mathbf{i}\neq\mathbf{j}\in\mathcal{I}_{\mathbf{n}}, \\ \varphi(0) &= 1,\ \varphi(-x)=\varphi(x),\ |\varphi(x)|\leq 1,\ \forall x, \end{aligned} \tag{8}$$

where $\varphi(x)\downarrow 0$, as $x\longrightarrow+\infty$. In addition, there exists two positive real numbers a, b such that

$$\mathbb{E}\left[\exp\left(a|\varepsilon_{\mathbf{i}}|^b\right)\right] < \infty. \tag{9}$$

Assumption 3. $\widehat{\lambda}_1 > \widehat{\lambda}_2 > \cdots > \widehat{\lambda}_{q_{\mathbf{n}}} > 0$ a.s.

The condition $\|X\|_G < M$ a.s. in Assumption 1 has been used in Mas and Pumo (2009); it is used to simplify the proofs and may be relaxed by assuming that $\sup_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}\|X_{\mathbf{i}}\|_G < M_{\mathbf{n}}$ a.s., where $\lim_{\mathbf{n}\to+\infty} M_{\mathbf{n}} = +\infty$ and $\lim_{\mathbf{n}\to+\infty}(M_{\mathbf{n}}^2\log\widehat{\mathbf{n}})/\widehat{\mathbf{n}}^{1/3} = 0$. In addition, if we consider a Fourier expansion of $X_{\mathbf{i}}$ in the orthonormal basis $\{v_j\}$ of G, that is $X_{\mathbf{i}}(t) = \sum_{j=1}^{+\infty}\langle X_{\mathbf{i}}, v_j\rangle v_j(t)$, then if the random fields $\zeta_j = \{\langle X_{\mathbf{i}}, v_j\rangle, \mathbf{i}\in\mathbb{Z}^d\}$ are α-mixing strictly stationary, Condition (7), weaker than mixing, is shown in Gneiting et al. (2010). Assumption 1 may be satisfied by spatial functional autoregressive processes. Condition (8) in Assumption 2 is satisfied by stationary Gaussian random fields (see, for instance, Francisco-Fernandez and Opsomer, 2005), whereas Condition (9) has been used in Biau and Cadre (2004). Assumption 3 is basic in functional data analysis. Assumption 3 has been used in Cardot et al. (2003).

2.3 Main asymptotic results

We give here the main asymptotic normality results of $\widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}}$.

Theorem 1. *Under Assumptions 1 and 2, we have under H_0*

$$\widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0,\sigma^2\Lambda^2\right),$$

where $\xrightarrow{\mathcal{D}}$ stands for the convergence in distribution and

$$\Lambda^2 = \sum_{j=1}^{+\infty}\lambda_j\lim_{\mathbf{n}\to\infty}\left(\frac{1}{\widehat{\mathbf{n}}}\sum_{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}}} h_j(\|\mathbf{i}-\boldsymbol{\ell}\|)\varphi(\|\mathbf{i}-\boldsymbol{\ell}\|)\right)\left\langle v_j,\cdot\right\rangle v_j.$$

The result below gives the asymptotic normality of the test statistic defined in (6).

Theorem 2. *Assume that Assumptions 1–3 hold and that there exists a sequence $(q_{\mathbf{n}})$ such that*

$$\frac{\widehat{\mathbf{n}}\lambda_{q_{\mathbf{n}}}^2}{\left(\sum_{j=1}^{q_{\mathbf{n}}}\beta_j\right)^2\log\widehat{\mathbf{n}}} \longrightarrow +\infty$$

as $\mathbf{n}\to+\infty$, where

$$\beta_j = \begin{cases} 2\sqrt{2}/(\lambda_1-\lambda_2) & \text{if } j=1, \\ 2\sqrt{2}/(\min(\lambda_{j-1}-\lambda_j, \lambda_j-\lambda_{j+1})) & \text{if } j\geq 2. \end{cases}$$

Then, under H_0, $S_{\mathbf{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,2)$.

The following result states the consistency of the testing proposed procedure.

Theorem 3. *Assume that Assumptions 1–3 hold with $\lambda_j = O(r^j)$, $0 < r < 1$, $j \geq 1$, then the testing procedure based on $S_{\mathbf{n}}$ is consistent.*

2.4 Case of the regression model with derivatives

This section generalises the previous results in the case of model (1). Let $\Gamma_{\mathbf{n}}$ and $\Delta_{\mathbf{n}}$ be the non-parametric empirical estimators of the operators Δ and Γ defined by $\Gamma_{\mathbf{n}} = \widehat{\mathbf{n}}^{-1} \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} X_{\mathbf{i}} \otimes_H X_{\mathbf{i}}$, $\Delta_{\mathbf{n}} x = \widehat{\mathbf{n}}^{-1} \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} Y_{\mathbf{i}} \langle X_{\mathbf{i}}, x\rangle_H$, $x \in H$. We denote by $(\widehat{\lambda}_j)_{j\geq 1}$ the sequence of eigenvalues of $\Gamma_{\mathbf{n}}$ associated to sequence of eigenfunctions $(\widehat{v}_j)_{j\geq 1}$. Assumptions and results in this section are obtained by replacing the inner product $\langle\cdot,\cdot\rangle$ in Sections 2.2 and 2.3 by $\langle\cdot,\cdot\rangle_H$. The following theorem extends the previous results to the case of model (1).

Theorem 4.

(i) *Under H_0 and Assumptions 1 and 2, we have*

$$\widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \sigma^2\Lambda^2\right).$$

(ii) *Under H_0 and the hypotheses of Theorem 2 with appropriate space and inner product, $S_{\mathbf{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 2)$.*

(iii) *Under H_1 and the hypotheses of Theorem 3 with appropriate space and inner product, the testing procedure is consistent.*

3. A simulation study

In this section, we illustrate our testing procedure by simulations. We computed empirical levels and powers of test from simulated spatial data in $\mathbb{Z}^2$. Using the lexico-graphic order, we simulated a sample $\{(X_{\mathbf{i}_\ell}, Y_{\mathbf{i}_\ell})\}_{1\leq \ell\leq n^2}$ as follows: we consider the 15-th first elements $B_1, \ldots, B_{15}$ of the Fourier basis. For $k = 1, \cdots, 15$, we simulated a pseudo-random vector $(\xi_{\mathbf{i}_1,k}, \ldots, \xi_{\mathbf{i}_{n^2},k})^T$ in $\mathbb{R}^{n^2}$ from the multivariate uniform distribution with null mean and covariance matrix Σ^1, where Σ^1 is the $n^2 \times n^2$ matrix with general term $\Sigma^1_{ij} = \exp(-\|\mathbf{i}_i - \mathbf{i}_j\|_2)$. For $\ell = 1, \ldots, n^2$, we take

$$X_{\mathbf{i}_\ell}(t) = \sum_{k=1}^{15} \xi_{\mathbf{i}_\ell,k}\, B_k(t).$$

The interval $[0; 1]$ is discretised by 366 equispaced points. We have then simulated $Y_{\mathbf{i}_\ell}$ by approximating integrals in equations (1) and (3) using the rectangular method. $\varepsilon_{\mathbf{i}_\ell}$ are i.i.d Gaussian random variables with zero mean and variance equals to σ^2. Two cases are considered in which, under H_0, $\sigma^2 = 0.01$.

Case A : Model (3) without the derivative. Under H_0, $\beta = 0$, and under H_1, $\beta(t) = [\sin(2\pi t^3)]^3$, if $t \in [0; 0.6]$ and $\beta(t) = 0$, if $t \in]0.6; 1]$. The interval $[0; 0.6]$ (resp. $]0.6; 1]$) is discretised by 7 (resp. 359) equispaced points. Under H_1, σ^2 is controlled by the signal-to-noise ratio (snr) defined

Table 1. **Cases A and B (with $\beta = 0$ and $\gamma = 0$).** Empirical levels (as percentage) of the test, based on 800 replications; $n^2 = 225$.

	$D_{\mathbf{n}}$			$S_{\mathbf{n}}$		
α (%)	$q_n = 5$	$q_n = 10$	$q_n = 20$	$q_n = 5$	$q_n = 10$	$q_n = 20$
20	33.6	29.6	7.5	15.1	16.8	33.8
10	19.0	16.0	2.3	9.5	8.5	14.6
5	11.6	8.6	0.8	6.6	4.5	5.0
1	2.6	1.5	0.1	2.8	1.8	0.1

by $snr1 = \mathbb{E}[\langle\beta, X\rangle^2]/(\mathbb{E}[\langle\beta, X\rangle^2] + \sigma^2)$. We consider the estimator $\widehat{\sigma}^2$ of σ^2 defined by

$$\widehat{\sigma}^2 = \frac{1}{tr(I-S)} \sum_{\ell=1}^{n^2} (Y_{\mathbf{i}_\ell} - \langle\widehat{\beta}, X_{\mathbf{i}_\ell}\rangle)^2, \tag{10}$$

where I is the $n^2 \times n^2$ identity matrix, S is the hat matrix, $tr(x)$ stands for the trace of the matrix x, and $\widehat{\beta}$ is an estimator of β computed by using the function "fregre.basis" of the R package fda.

Case B : Model (1) with the derivative. Under H_0, $\beta = \gamma = 0$, and under H_1, $\beta(t) = [\sin(2\pi t^3)]^3$ if $t \in [0; 0.6]$ and $\beta(t) = 0$ if $t \in]0.6; 1]$, and $\gamma(t) = (0.6-t)^2$ if $t \in [0; 0.6]$ and $\gamma(t) = 0$ if $t \in]0.6; 1]$. The interval $[0; 0.6]$ (resp. $]0.6; 1]$) is discretised by 7 (resp. 359) equispaced points. Under H_1, σ^2 is controlled by the signal-to-noise ratio (snr) defined by

$$snr2 = \frac{\mathbb{E}((\langle\beta, X_{\mathbf{i}}\rangle_H + \langle\gamma, X'_{\mathbf{i}}\rangle)^2)}{\mathbb{E}((\langle\beta, X_{\mathbf{i}}\rangle_H + \langle\gamma, X'_{\mathbf{i}}\rangle)^2) + \sigma^2},$$

where $X'_{\mathbf{i}}$ is computed by using the function "fdata.deriv" of the R package fda. We consider the estimator $\widehat{\sigma}^2$ of σ^2 defined in (10) and in which we replace $\langle\widehat{\beta}, X_{\mathbf{i}}\rangle$ by $\widehat{\Psi}(X_{\mathbf{i}}) = \langle\widehat{\beta} + D^*\widehat{\beta}' + D^*\widehat{\gamma}, X_{\mathbf{i}}\rangle$. The eigenelements of the covariance operator are estimated by means of a quadrature rule. We assess performance of our testing procedure through empirical levels and powers of test, based on 800 replications with $n^2 = 225$.

In Table 1, we remark that for larger value of q_n (i.e $q_n = 20$), the empirical levels based on $S_{\mathbf{n}}$ overestimate nominal levels $\alpha = 20\%,\ 10\%,\ 5\%$, and it is contrary for those based on $D_{\mathbf{n}}$. Whereas for $q_n = 5,\ 10$, the empirical levels computed from $S_{\mathbf{n}}$ tend to underestimate nominal levels $\alpha = 20\%,\ 10\%,\ 5\%$, and it is contrary for those computed from $D_{\mathbf{n}}$. However, for $q_n = 10$, compared to the test statistic $D_{\mathbf{n}}$, the empirical levels obtained from $S_{\mathbf{n}}$ lead to better approximations of the true nominal levels $\alpha = 20\%,\ 10\%,\ 5\%$. So, we analyse the empirical power whose values are given in the following Tables 2 and 3.

In Tables 2 and 3, we remark that empirical powers increase as the signal-to-noise ratio increases and decrease as q_n increases. So, the both proposed test statistic computed with $q_n = 5$, is more power to reject the null hypothesis H_0 when it is false. Thus, a trade off between these two risks is needed to choose a value of q_n before deciding to accept or to reject the null hypothesis. In this case of this simulation study, we can choose $q_n = 10$.

Table 2. **Case A.** The empirical powers (as percentage) of the test, based on 800 replications; $n^2 = 225$.

		$D_{\mathbf{n}}$			$S_{\mathbf{n}}$		
$snr1$(%)	α (%)	$q_n = 5$	$q_n = 10$	$q_n = 20$	$q_n = 5$	$q_n = 10$	$q_n = 20$
5	1	64.1	46.5	15.3	64.5	47.1	15.1
	5	82.5	70.4	32.9	74.8	61.6	27.5
	10	90.1	80.9	44.0	79.5	68.4	33.4
10	1	96.1	89.1	58.9	96.1	89.5	58.6
	5	99.0	96.0	78.8	97.8	93.5	71.9
	10	99.8	98.5	86.1	98.9	95.6	78.6

Table 3. **Case B.** The empirical powers (as percentage) of the test, based on 800 replications; $n^2 = 225$.

		$D_{\mathbf{n}}$			$S_{\mathbf{n}}$		
$snr2$(%)	α (%)	$q_n = 5$	$q_n = 10$	$q_n = 20$	$q_n = 5$	$q_n = 10$	$q_n = 20$
5	1	56.5	44.6	15.5	57.3	45.4	15.3
	5	77.8	66.5	34.5	69.0	58.0	25.8
	10	87.6	78.6	46.3	74.1	65.1	33.6
10	1	93.6	87.3	60.6	93.8	87.6	60.5
	5	98.9	96.1	79.0	96.6	93.5	72.0
	10	99.8	98.6	87..4	98.0	95.4	78.4

4. Conclusion

In this work, we propose to test whether or not the functional linear relationship suggested by model (1) and its particular case (3) hold. The originality of the proposed method is to consider spatially dependent data and to take into account a first derivative effect on the functional linear model. The difficulty involved in handling the spatial dependency is in the proof of the asymptotic normality of the empirical cross-covariance operator. We propose two statistics which take into account the proximity between sites. Then, we established their asymptotic distribution. The simulation study showed that the testing procedure based on both statistics $S_{\mathbf{n}}$ and $D_{\mathbf{n}}$ give good results. Notice that, from Proposition 1, the case B with derivatives can be rewritten as case A without derivatives of functional data. So, the proposed methodology can, therefore, be seen as a good alternative to Cardot et al. (2003) when available data are spatially dependent with or without a derivative effect in the functional linear model.

5. Proofs

For proving Theorem 1, we need the following lemmas on the asymptotic variance and characteristic function of $\langle \widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}}, x\rangle_{\mathcal{HS}}$.

Lemma 1. *Under H_0 and Assumptions 1 and 2, for all x in G and as $\mathbf{n} \to +\infty$, we have*

$$Var\left(\left\langle \widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}}, x\right\rangle_{\mathcal{HS}}\right) \longrightarrow \sigma^2 \sum_{j=1}^{+\infty} \lambda_j \lim_{\mathbf{n}\to\infty} \left(\frac{1}{\widehat{\mathbf{n}}} \sum_{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}}} h_j(\|\mathbf{i}-\boldsymbol{\ell}\|)\varphi(\|\mathbf{i}-\boldsymbol{\ell}\|)\right) \left\langle v_j, x\right\rangle^2.$$

Proof. Since $\{X_{\mathbf{i}}\}$ is strictly stationary, then from Assumptions 1 and 2, we have

$$\begin{aligned}
A_{\mathbf{n}} := \; & Var\left(\left\langle \widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}}, x\right\rangle_{\mathcal{HS}}\right) = \mathbb{E}\left\{\left[\sum_{j=1}^{+\infty} \frac{\widehat{\mathbf{n}}^{1/2}}{\widehat{\mathbf{n}}} \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} \langle X_{\mathbf{i}}, v_j\rangle\langle x, v_j\rangle\varepsilon_{\mathbf{i}}\right]^2\right\} \\
= \; & \frac{1}{\widehat{\mathbf{n}}} \sum_{j=1}^{+\infty} \langle x, v_j\rangle^2 \sum_{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}}} \mathbb{E}\left[\langle X_{\mathbf{i}}, v_j\rangle\langle X_{\boldsymbol{\ell}}, v_j\rangle\right] Cov\left(\varepsilon_{\mathbf{i}}, \varepsilon_{\boldsymbol{\ell}}\right) \\
& + \frac{1}{\widehat{\mathbf{n}}} \sum_{\substack{j,k=1 \\ j\neq k}}^{+\infty} \langle x, v_j\rangle\langle x, v_k\rangle \sum_{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}}} \mathbb{E}\left[\langle X_{\mathbf{i}}, v_j\rangle\langle X_{\boldsymbol{\ell}}, v_k\rangle\right] Cov\left(\varepsilon_{\mathbf{i}}, \varepsilon_{\boldsymbol{\ell}}\right) \\
= \; & \sigma^2 \sum_{j=1}^{+\infty} \lambda_j \left(\frac{1}{\widehat{\mathbf{n}}} \sum_{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}}} h_j(\|\mathbf{i}-\boldsymbol{\ell}\|)\varphi(\|\mathbf{i}-\boldsymbol{\ell}\|)\right) \left\langle v_j, x\right\rangle^2.
\end{aligned}$$

However, for all $j = 1, 2, \cdots$, we have

$$\begin{aligned}
A_{j\mathbf{n}} \; &:= \; \frac{1}{\widehat{\mathbf{n}}} \sum_{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}}} h_j(\|\mathbf{i}-\boldsymbol{\ell}\|)\varphi(\|\mathbf{i}-\boldsymbol{\ell}\|) \\
&\leq \; \frac{1}{\widehat{\mathbf{n}}} \sum_{k=0}^{+\infty} \sum_{\substack{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}} \\ k\leq\|\mathbf{i}-\boldsymbol{\ell}\|<k+1}} h_j(\|\mathbf{i}-\boldsymbol{\ell}\|)\varphi(\|\mathbf{i}-\boldsymbol{\ell}\|) \\
&\leq \; \sum_{k=0}^{+\infty} \sum_{\substack{\mathbf{i}\in\mathcal{I}_{\mathbf{n}} \\ k\leq\|\mathbf{i}\|=t<k+1}} h_j(t)\varphi(t) \leq \sum_{t=1}^{+\infty} t^{d-1} h_j(t)\varphi(t).
\end{aligned}$$

Then, from Assumptions 1 and 2, we have

$$A_{\mathbf{n}} \leq \sigma^2 \sum_{j=1}^{+\infty} \lambda_j \sum_{t=1}^{+\infty} t^{d-1} h_j(t)\varphi(t) \left\langle v_j, x\right\rangle^2 \leq \sigma^2 \|x\|_G^2 \sum_{t=1}^{+\infty} t^{d-1} \sum_{j=1}^{+\infty} \lambda_j h_j(t) < \infty.$$

So, there exists some positive constant E such that $\lim_{\mathbf{n}\to\infty} A_{\mathbf{n}} = E < \infty$. ■

Let us consider $\Xi_{\mathbf{i}} x = Z_{\mathbf{i}}(x)/\widehat{\mathbf{n}}$, where $Z_{\mathbf{i}}(x) = \langle X_{\mathbf{i}}, x\rangle_{\mathcal{HS}}\, \varepsilon_{\mathbf{i}}$. Then, under H_0, we have

$$\langle \Delta_{\mathbf{n}}, x\rangle_{\mathcal{HS}} = \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} \Xi_{\mathbf{i}} x = \sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} \frac{Z_{\mathbf{i}}(x)}{\widehat{\mathbf{n}}}.$$

Let us use the Bernstein's spatial blocks technique (Tran, 1990) to introduce the test statistics. Assume, without loss of generality, that there exist integers u_i, $i = 1, \cdots, d$, p_1 and p_2 in $\mathbb{N}^*$ such that $n_i = u_i(p_1+p_2)$, $\widehat{t} = u_1 \times \cdots \times u_d$, and $\widehat{\mathbf{n}} = \widehat{t}(p_1+p_2)^d$. For $\mathbf{l} = (l_1, ..., l_d)$, $l_i \in \{0, ..., u_i - 1\}$, $i = 1, \cdots, d$ let

$$\begin{aligned}
U(1,\mathbf{n},\mathbf{l},x) &= \sum_{\substack{i_k = l_k(p_1+p_2)+1 \\ k=1,\dots,d}}^{l_k(p_1+p_2)+p_1} \Xi_{\mathbf{i}} x, \\
U(2,\mathbf{n},\mathbf{l},x) &= \sum_{\substack{i_k = l_k(p_1+p_2)+1 \\ k=1,\dots,d-1}}^{l_k(p_1+p_2)+p_1} \sum_{i_d = l_d(p_1+p_2)+p_1+1}^{(l_d+1)(p_1+p_2)} \Xi_{\mathbf{i}} x, \\
U(3,\mathbf{n},\mathbf{l},x) &= \sum_{\substack{i_k = l_k(p_1+p_2)+1 \\ k=1,\dots,d-2}}^{l_k(p_1+p_2)+p_1} \sum_{i_{d-1} = l_{d-1}(p_1+p_2)+p_1+1}^{(l_{d-1}+1)(p_1+p_2)} \sum_{i_d = l_d(p_1+p_2)+1}^{l_d(p_1+p_2)+p_1} \Xi_{\mathbf{i}} x, \\
U(4,\mathbf{n},\mathbf{l},x) &= \sum_{\substack{i_k = l_k(p_1+p_2)+1 \\ k=1,\dots,d-2}}^{l_k(p_1+p_2)+p_1} \sum_{i_{d-1} = l_{d-1}(p_1+p_2)+p_1+1}^{(l_{d-1}+1)(p_1+p_2)} \sum_{i_d = l_d(p_1+p_2)+1}^{(l_d+1)(p_1+p_2)} \Xi_{\mathbf{i}} x,
\end{aligned}$$

and so on, until the last two terms

$$\begin{aligned}
U(2^{d-1}+1,\mathbf{n},\mathbf{l},x) &= \sum_{\substack{i_k = l_k(p_1+q_1)+p_1+1 \\ k=1,\dots,d-1}}^{(l_k+1)(p_1+p_2)} \sum_{i_d = l_d(p_1+p_2)+1}^{l_d(p_1+p_2)+p_1} \Xi_{\mathbf{i}} x, \\
U(2^d,\mathbf{n},\mathbf{l},x) &= \sum_{\substack{i_k = l_k(p_1+p_2)+p_1+1 \\ k=1,\dots,d}}^{(l_k+1)(p_1+p_2)} \Xi_{\mathbf{i}} x.
\end{aligned}$$

Setting $\mathcal{T} = \{0, ..., u_1 - 1\} \times \cdots \times \{0, ..., u_d - 1\}$, let for each integer $i = 1, ..., 2^d$,

$$T(\mathbf{n}, i, x) = \sum_{\mathbf{j} \in \mathcal{T}} U(i, \mathbf{n}, \mathbf{j}, x).$$

Then, under H_0, $\langle \Delta_{\mathbf{n}}, x \rangle_{\mathcal{HS}}$ may be rewritten as

$$\langle \Delta_{\mathbf{n}}, x \rangle_{\mathcal{HS}} = \sum_{i=1}^{2^d} T(\mathbf{n}, i, x).$$

We have the following:

Lemma 2. *Under H_0 and Assumptions 1 and 2, as $\mathbf{n} \longrightarrow +\infty$ we have:*

i) For all $x \in W$,

$$Q1 = \mathbb{E}\left[\exp\left(i\ \widehat{\mathbf{n}}^{1/2} T(\mathbf{n}, 1, x)\right)\right] - \Pi_{\mathbf{l} \in \mathcal{T}} \mathbb{E}\left[\exp\left(i\ \ \widehat{\mathbf{n}}^{1/2} U(1, \mathbf{n}, \mathbf{l}, x)\right)\right] \longrightarrow 0,$$

where $i^2 = -1$;

ii) $Q2 = \ \widehat{\mathbf{n}} \mathbb{E}\left[\left(\sum_{i=2}^{2^d} T(\mathbf{n}, i, x)\right)^2\right] \longrightarrow 0$;

iii) $Q3 = \widehat{\mathbf{n}} \sum_{\mathbf{l} \in \mathcal{T}} \mathbb{E}\left[(U(1,\mathbf{n},\mathbf{l},x))^2\right] \longrightarrow \sigma^2 \sum_{j=1}^{+\infty} \lambda_j E_j \left\langle v_j, x\right\rangle^2$, *where*

$$E_j = \lim_{\mathbf{n}\to\infty} \left(\frac{1}{\widehat{\mathbf{n}}} \sum_{\mathbf{i},\ell \in \mathcal{I}_{\mathbf{n}}} h_j(\|\mathbf{i}-\ell\|)\varphi(\|\mathbf{i}-\ell\|) \right);$$

iv) $Q4 = \widehat{\mathbf{n}} \sum_{\mathbf{l} \in \mathcal{T}} \mathbb{E}\left[(U(1,\mathbf{n},\mathbf{l},x))^2 \boldsymbol{I}_{\{|U(1,\mathbf{n},\mathbf{l},x)| > \tau \widehat{\mathbf{n}}^{-1/2}\}}\right] \longrightarrow 0$, *for all* $\tau > 0$.

Proof. Let $p_1 = p_1(\mathbf{n})$ and $p_2 = p_2(\mathbf{n})$, $p_1 = \left\lfloor \widehat{\mathbf{n}}^{1/(3d)} \right\rfloor$, $p_2 = o\left(\widehat{\mathbf{n}}^{1/(4d)}\right)$, then

$$\frac{p_2}{p_1} \leq \frac{C}{\widehat{\mathbf{n}}^{1/(12d)}} \longrightarrow 0,$$

$p_2 < p_1$, asymptotically.

Proof of (i). Sort the random variables $U(1,\mathbf{n},\mathbf{l},x)$ and refer to them as $\widetilde{U}_1$,...,$\widetilde{U}_{\widehat{t}}$. Recall that $Card(\mathcal{T}) := \widehat{t} = \widehat{\mathbf{n}}(p_1+p_2)^{-d} \leq \widehat{\mathbf{n}} p_1^{-d}$. Denote by $I(1,\mathbf{n},\mathbf{l}) = \{\mathbf{i} : l_k(p_1+p_2)+1 \leq i_k \leq l_k(p_1+p_2)+p_1\}$, the set of sites involved with $U(1,\mathbf{n},\mathbf{l},x)$ and $\widetilde{I}_j$ the set of sites involved with $\widetilde{U}_j$. Lemma 3.1 of Tran (1990) gives

$$\begin{aligned} Q1 \quad &\leq \quad \sum_{k=1}^{\widehat{t}-1} \sum_{j=k+1}^{\widehat{t}} \Bigg| \mathbb{E}(\exp(i\widehat{\mathbf{n}}^{1/2}\widetilde{U}_k) - 1)(\exp(i\widehat{\mathbf{n}}^{1/2}\widetilde{U}_j) - 1) \prod_{s=j+1}^{\widehat{t}} \exp(i\widehat{\mathbf{n}}^{1/2}\widetilde{U}_s) \\ &\qquad - \mathbb{E}(\exp(i\widehat{\mathbf{n}}^{1/2}\widetilde{U}_k) - 1)\mathbb{E}(\exp(i\widehat{\mathbf{n}}^{1/2}\widetilde{U}_j) - 1) \prod_{s=j+1}^{\widehat{t}} \exp(i\widehat{\mathbf{n}}^{1/2}\widetilde{U}_s) \Bigg| \\ &\qquad := \sum_{k=1}^{\widehat{t}-1} \sum_{j=k+1}^{\widehat{t}} Q_{kj}. \end{aligned}$$

In addition, by Lemma 2.1 ii) of Tran (1990) $Q_{kj} \leq c\alpha_{1,\infty}(\rho(\widetilde{I}_j, \widetilde{I}_k))p_1^d$. Since $\xi > 4d$, then

$$\begin{aligned} Q1 \quad &\leq \quad c p_1^d \sum_{k=1}^{\widehat{t}-1} \sum_{j=k+1}^{\widehat{t}} \alpha_{1,\infty}(\rho(\widetilde{I}_j, \widetilde{I}_k)) \leq c\widehat{\mathbf{n}} \sum_{j=1}^{+\infty} j^{d-1} \alpha_{1,\infty}(jp_2) \\ &\leq \quad c_1 \widehat{\mathbf{n}}^{-(\xi-4d)/(4d)} \longrightarrow 0. \end{aligned}$$

Proof of (ii). It is equivalent to show that

$$\widehat{\mathbf{n}}\mathbb{E}\left[(T(\mathbf{n},i,x))^2\right] \longrightarrow 0 \quad \text{for each} \quad 2 \leq i \leq 2^d.$$

Sort the random variables $U(2,\mathbf{n},\mathbf{l},x)$ and refer to them as $\widehat{U}_1$,...,$\widehat{U}_{\widehat{t}}$ and let

$$\mathbb{E}\left[(T(\mathbf{n},2,x))^2\right] \quad = \quad \sum_{j=1}^{\widehat{t}} Var(\widehat{U}_i) + 2\sum_{i=1}^{\widehat{t}} \sum_{\substack{j=1 \\ i>j}}^{\widehat{t}} Cov(\widehat{U}_i, \widehat{U}_j) := \widetilde{A}_1 + \widetilde{A}_2.$$

Under Assumptions 1 and 2, we have

$$\begin{aligned}
Var(\widehat{U}_i) &= Var\left(\sum_{\substack{i_k=1\\k=1,\cdots,d-1}}^{p_1}\sum_{i_d=1}^{p_2}\Xi_{\mathbf{i}\boldsymbol{\delta}}x\right)\\
&\leq \frac{\sigma^2 p_1^{d-1}p_2}{\widehat{\mathbf{n}}^2}\left(\sum_{k=1}^{+\infty}\lambda_k\langle x,v_k\rangle^2+\sum_{k=1}^{+\infty}\lambda_k\sum_{\substack{l_k=1\\k=1,\dots,d-1}}^{p_1}\sum_{l_d=1}^{p_2}h_k(\|\mathbf{l}\|)\varphi(\|\mathbf{l}\|)\langle x,v_k\rangle^2\right)\\
&\leq \frac{\|x\|_G^2\sigma^2p_1^{d-1}p_2}{\widehat{\mathbf{n}}^2}\left(\sum_{k=1}^{+\infty}\lambda_k+\sum_{t=1}^{+\infty}t^{d-1}\sum_{k=1}^{+\infty}\lambda_kh_k(t)\right)\leq\frac{Cp_1^{d-1}p_2}{\widehat{\mathbf{n}}^2}.
\end{aligned}$$

Then, we have

$$\widehat{\mathbf{n}}\widetilde{A}_1\leq Cp_1^{d-1}p_2(p_1+p_2)^{-d}\leq C\left(\frac{p_2}{p_1}\right)\longrightarrow 0.$$

Let

$$\begin{aligned}
I(2,\mathbf{n},\mathbf{l}) &= \{\mathbf{i}: l_m(p_1+p_2+1)\leq i_m\leq l_m(p_1+p_2)+p_1, 1\leq m\leq d-1,\\
&\quad l_d(p_1+p_2)+p_1+1\leq i_d\leq(l_d+1)(p_1+p_2)\},
\end{aligned}$$

then $U(2,\mathbf{n},\mathbf{l},x)$ is the sum of $\Xi_{\mathbf{i}}x$ over all sites in $I(2,\mathbf{n},\mathbf{l})$. Since $p_1>p_2$, if $\mathbf{l}$ and $\mathbf{l}'$ belong to two distinct sets $I(2,\mathbf{n},\mathbf{l})$ and $I(2,\mathbf{n},\mathbf{l}')$, then $l_m\neq l'_m$ for some $1\leq m\leq d$ and $\|\mathbf{l}-\mathbf{l}'\|>p_2$, and by Assumptions 1 and 2, we have

$$\begin{aligned}
\widehat{\mathbf{n}}|\widetilde{A}_2| &\leq c\widehat{\mathbf{n}}\sum_{\substack{i_m=1\\m=1,\dots,d}}^{n_m}\sum_{\substack{l_m=1\\m=1,\dots,d\\\|\mathbf{i}-\mathbf{l}\|>p_2}}^{n_m}|Cov(\Xi_{\mathbf{i}}x,\Xi_{\mathbf{l}}x)|\\
&\leq C\varphi(p_2)\sum_{t=p_2+1}^{+\infty}t^{d-1}\sum_{k=1}^{+\infty}\lambda_kh_k(t)\longrightarrow 0. \qquad (11)
\end{aligned}$$

Proof of (iii). We have

$$\langle\Delta_{\mathbf{n}},x\rangle_{\mathcal{HS}}=\sum_{i=1}^{2^d}T(\mathbf{n},i,x)=T(\mathbf{n},1,x)+\sum_{i=2}^{2^d}T(\mathbf{n},i,x)=S'_{\mathbf{n}}+S''_{\mathbf{n}}.$$

From Lemma 1, we have

$$\mathbb{E}\left(\left\langle\widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}},x\right\rangle_{\mathcal{HS}}^2\right)\longrightarrow\sigma^2\sum_{j=1}^{+\infty}\lambda_j\lim_{\mathbf{n}\to\infty}\left(\frac{1}{\widehat{\mathbf{n}}}\sum_{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}}}h_j(\|\mathbf{i}-\boldsymbol{\ell}\|)\varphi(\|\mathbf{i}-\boldsymbol{\ell}\|)\right)\left\langle v_j,x\right\rangle^2.$$

This combined with Lemma 2(ii), gives

$$\widehat{\mathbf{n}}\mathbb{E}\left[(S'_{\mathbf{n}})^2\right]\longrightarrow\sigma^2\sum_{j=1}^{+\infty}\lambda_j\lim_{\mathbf{n}\to\infty}\left(\frac{1}{\widehat{\mathbf{n}}}\sum_{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}}}h_j(\|\mathbf{i}-\boldsymbol{\ell}\|)\varphi(\|\mathbf{i}-\boldsymbol{\ell}\|)\right)\left\langle v_j,x\right\rangle^2.$$

In addition, we have

$$\widehat{\mathbf{n}}\mathbb{E}\left[\left(S'_{\mathbf{n}}\right)^2\right] = \widehat{\mathbf{n}}\sum_{\mathbf{l}\in\mathcal{T}}\mathbb{E}[U(1,\mathbf{n},\mathbf{l},x)^2] + \widehat{\mathbf{n}}\sum_{\substack{\mathbf{l},\mathbf{l}'\in\mathcal{T}\\ \mathbf{l}\neq\mathbf{l}'}} Cov[U(1,\mathbf{n},\mathbf{l},x),U(1,\mathbf{n},\mathbf{l}',x)].$$

However, by the same arguments as those used at the end of the proof of $Q2$ in (ii), we have

$$\widehat{\mathbf{n}}\sum_{\substack{\mathbf{l},\mathbf{l}'\in\mathcal{T}\\ \mathbf{l}\neq\mathbf{l}'}} Cov[U(1,\mathbf{n},\mathbf{l},x),U(1,\mathbf{n},\mathbf{l}',x)] \longrightarrow 0.$$

Thus

$$\widehat{\mathbf{n}}\sum_{\mathbf{l}\in\mathcal{T}}\mathbb{E}[U(1,\mathbf{n},\mathbf{l},x)^2] \to \sigma^2\sum_{j=1}^{+\infty}\lambda_j\lim_{\mathbf{n}\to\infty}\left(\frac{1}{\widehat{\mathbf{n}}}\sum_{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}}} h_j(\|\mathbf{i}-\boldsymbol{\ell}\|)\varphi(\|\mathbf{i}-\boldsymbol{\ell}\|)\right)\left\langle v_j,x\right\rangle^2. \tag{12}$$

Proof of (iv). Since $X_{\mathbf{i}\delta}$ is almost surely bounded and $\varepsilon_{\mathbf{i}\delta}$ is not bounded, we use the following truncation argument: $Z_{\mathbf{i}}^{\star}(x) = \langle X_{\mathbf{i}},x\rangle_{\mathcal{HS}}\,\varepsilon_{\mathbf{i}}\mathbf{1}_{|\varepsilon_{\mathbf{i}}|\leq L}$ and $Z_{\mathbf{i}}^{\star\star}(x) = \langle X_{\mathbf{i}},x\rangle_{\mathcal{HS}}\,\varepsilon_{\mathbf{i}}\mathbf{1}_{|\varepsilon_{\mathbf{i}}|> L}$, where L is some positive constant such that $L^2/\widehat{\mathbf{n}}^{1/3}\to 0$ as $\mathbf{n}\to+\infty$. So, we have

$$|\Xi_{\mathbf{i}}^{\star}x| = \frac{Z_{\mathbf{i}}^{\star}(x)}{\widehat{\mathbf{n}}} \leq \frac{cL}{\widehat{\mathbf{n}}} \quad \text{a.s.},$$

where c is a positive constant. With the choices $p_1 = \left\lfloor\widehat{\mathbf{n}}^{1/(3d)}\right\rfloor$ then $\sqrt{\widehat{\mathbf{n}}}\,|U^{\star}(1,\mathbf{n},\mathbf{l},x)| \leq cp_1^dL/\sqrt{\widehat{\mathbf{n}}} \leq cL/\widehat{\mathbf{n}}^{1/6} \longrightarrow 0$ a.s. Thus for all $\mathbf{l}$, $\tau>0$ and for $\mathbf{n}$ large enough, we have

$$Q4^{\star} \leq \frac{C_1L^2}{\widehat{\mathbf{n}}^{1/3}}\sum_{\substack{l_k=1\\ k=1,\dots,d}}^{u_k-1}\mathbb{P}\left(|U^{\star}(1,\mathbf{n},\mathbf{l},x)| > \tau^2\widehat{\mathbf{n}}^{-1/2}\right) = 0,$$

where C_1 is some positive constant. Then Lemma 1 holds with

$$V^{\star}(x) = \sum_{j=1}^{+\infty}\lambda_j\lim_{\mathbf{n}\to\infty}\left(\frac{1}{\widehat{\mathbf{n}}}\sum_{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}}} h_j(\|\mathbf{i}-\boldsymbol{\ell}\|)Cov(\varepsilon_{\mathbf{i}}\mathbf{1}_{|\varepsilon_{\mathbf{i}}|\leq L},\varepsilon_{\boldsymbol{\ell}}\mathbf{1}_{|\varepsilon_{\boldsymbol{\ell}}|\leq L})\right)\left\langle v_j,x\right\rangle^2,$$

and $\Delta_{\mathbf{n}}^{\star}$ which is obtained by replacing $\varepsilon_{\mathbf{i}}$ into $\Delta_{\mathbf{n}}$ by $\varepsilon_{\mathbf{i}}\mathbf{1}_{|\varepsilon_{\mathbf{i}}|\leq L}$. Therefore

$$\left\langle\widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}}^{\star},x\right\rangle_{\mathcal{HS}} \to \mathcal{N}(0,V^{\star}(x)) \text{ as } \mathbf{n}\to+\infty. \tag{13}$$

Since $\Delta_{\mathbf{n}} = \Delta_{\mathbf{n}}^{\star}+\Delta_{\mathbf{n}}^{\star\star}$, and putting

$$V(x) = \sum_{j=1}^{+\infty}\lambda_j\lim_{\mathbf{n}\to\infty}\left(\frac{1}{\widehat{\mathbf{n}}}\sum_{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}}} h_j(\|\mathbf{i}-\boldsymbol{\ell}\|)Cov(\varepsilon_{\mathbf{i}},\varepsilon_{\boldsymbol{\ell}})\right)\left\langle v_j,x\right\rangle^2,$$

we have

$$\begin{aligned}
&\left|\mathbb{E}\left[\exp\left(iu\left\langle\widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}},x\right\rangle_{\mathcal{HS}}\right)\right]-\exp\left(-u^2V(x)/2\right)\right|\\
\leq\ &\left|\mathbb{E}\left[\exp\left(iu\left\langle\widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}}^{\star},x\right\rangle_{\mathcal{HS}}\right)\right]-\exp\left(-u^2V^{\star}(x)/2\right)\right|\\
+\ &\left|\mathbb{E}\left[\exp\left(iu\left\langle\widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}}^{\star\star},x\right\rangle_{\mathcal{HS}}\right)\right]-1\right|+\left|\exp\left(-u^2V^{\star}(x)/2\right)-\exp\left(-u^2V(x)/2\right)\right|.
\end{aligned}$$

However, from the relation in (13), we have

$$\left|\mathbb{E}\left[\exp\left(iu\left\langle\widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}}^{\star},x\right\rangle_{\mathcal{HS}}\right)\right]-\exp\left(-u^2V^{\star}(x)/2\right)\right|\longrightarrow 0\ \text{ as }\ \mathbf{n}\to+\infty,$$

and since from the dominated convergence theorem, $V^{\star}(x)\to V(x)$ as $L\to+\infty$, we have

$$\left|\exp\left(-u^2V^{\star}(x)/2\right)-\exp\left(-u^2V(x)/2\right)\right|\longrightarrow 0\ \text{ as }\ L\to+\infty.$$

Besides, by arguments of Lemma 1, we have

$$\begin{aligned}
Var\left(\left\langle\widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}}^{\star\star},x\right\rangle_{\mathcal{HS}}\right) &= \sum_{j=1}^{+\infty}\lambda_j\left(\frac{1}{\widehat{\mathbf{n}}}\sum_{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}}}h_j(\|\mathbf{i}-\boldsymbol{\ell}\|)Cov(\varepsilon_{\mathbf{i}}\mathbf{1}_{|\varepsilon_{\mathbf{i}}|>L},\varepsilon_{\boldsymbol{\ell}}\mathbf{1}_{|\varepsilon_{\boldsymbol{\ell}}|>L})\right)\left\langle v_j,x\right\rangle^2\\
&\le \sum_{j=1}^{+\infty}\lambda_j\left(\frac{1}{\widehat{\mathbf{n}}}\sum_{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}}}h_j(\|\mathbf{i}-\boldsymbol{\ell}\|)\mathbb{E}(\varepsilon_{\mathbf{i}}^2\mathbf{1}_{|\varepsilon_{\mathbf{i}}|>L})\right)\left\langle v_j,x\right\rangle^2\\
&\le \|x\|_G^2\sum_{t=1}^{+\infty}t^{d-1}\sum_{j=1}^{+\infty}\lambda_jh_j(t)\left[\mathbb{E}\left(\varepsilon_{\mathbf{i}}^4\right)\right]^{1/2}\left[\mathbb{P}\left(|\varepsilon_{\mathbf{i}}|>L\right)\right]^{1/2}\\
&\le C_2\left[\frac{\mathbb{E}\left[\exp\left(a|\varepsilon_{\mathbf{i}}|^b\right)\right]}{\exp(aL^b)}\right]^{1/2},
\end{aligned}$$

where C_2 is some positive constant, then $\left\langle\widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}}^{\star\star},x\right\rangle_{\mathcal{HS}}=O_{a.s}\left(\exp(-aL^b/2)\right)$. Therefore

$$\left|\mathbb{E}\left[\exp\left(iu\left\langle\widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}}^{\star\star},x\right\rangle_{\mathcal{HS}}\right)\right]-1\right|\longrightarrow 0\ \text{ as }\ L\to+\infty.$$ ■

Proof of Theorem 1

Let $\langle V_{\mathbf{n}},x\rangle_{\mathcal{HS}}:=\left\langle\widehat{\mathbf{n}}^{1/2}\Delta_{\mathbf{n}},x\right\rangle_{\mathcal{HS}}=\widehat{\mathbf{n}}^{1/2}T(\mathbf{n},1,x)+\widehat{\mathbf{n}}^{1/2}\sum_{i=2}^{2^d}T(\mathbf{n},i,x)$. From Lemma 2 ii), the rightmost term converges in probability towards zero. Moreover, from Lemma 2 i), the random variables $(U(1,\mathbf{n},\mathbf{l},x),\mathbf{l}\in\mathcal{T})$ are asymptotically independent. Then, from Lemma 2 iii), iv) and Slutsky's Theorem, we obtain, for all x in W, $\mathbb{E}\left[\exp\left(iu\left\langle V_{\mathbf{n}},x\right\rangle_{\mathcal{HS}}\right)\right]\longrightarrow\mathbb{E}\left[\exp\left(iu\langle C_\Delta,x\rangle_{\mathcal{HS}}\right)\right]$, where $i^2=-1$, and $C_\Delta\sim\mathcal{N}(0,\sigma^2\Lambda^2)$. Next, we show that $\left(\mathbb{P}_{V_{\mathbf{n}}}\right)$ is tight. Since Lemma 1 gives

$$Var(\langle V_{\mathbf{n}},x\rangle_{\mathcal{HS}})\to\sigma^2\sum_{j=1}^{+\infty}\lambda_j\lim_{\mathbf{n}\to\infty}\left(\frac{1}{\widehat{\mathbf{n}}}\sum_{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}}}h_j(\|\mathbf{i}-\boldsymbol{\ell}\|)\varphi(\|\mathbf{i}-\boldsymbol{\ell}\|)\right)\left\langle v_j,x\right\rangle^2,$$

then

$$R_t^2=\mathbb{E}\left(\sum_{j=t}^{+\infty}\left\langle V_{\mathbf{n}},v_j\right\rangle^2\right)\longrightarrow\sigma^2\sum_{j=t}^{+\infty}\lambda_j\lim_{\mathbf{n}\to\infty}\left(\frac{1}{\widehat{\mathbf{n}}}\sum_{\mathbf{i},\boldsymbol{\ell}\in\mathcal{I}_{\mathbf{n}}}h_j(\|\mathbf{i}-\boldsymbol{\ell}\|)\varphi(\|\mathbf{i}-\boldsymbol{\ell}\|)\right)<+\infty.$$

Given $\zeta>0$, consider $\Lambda_\zeta=\bigcap_{k=1}^{+\infty}A_k$ where $A_k=\{x\in W:\sum_{j=t_k}^{+\infty}\left\langle x,v_j\right\rangle^2\le M_k^{-1}\}$, with $1=t_1<t_2<\ldots<t_k\ldots$, $M_k\to+\infty$ as $k\to+\infty$ and $\sum_{k=1}^{+\infty}M_kR_{t_k}^2<\zeta$. Then, from Tchebytchev's inequality, we have

$$1-\mathbb{P}_{V_{\mathbf{n}}}\left(\Lambda_\zeta\right)\le\sum_{k=1}^{+\infty}\mathbb{P}_{V_{\mathbf{n}}}\left(A_k^c\right)\le\sum_{k=1}^{+\infty}M_kR_{t_k}^2<\zeta.$$

Thus $\left(\mathbb{P}_{V_{\mathbf{n}}}\right)$ is tight. Finally, applying Dudley-Skorokhod's theorem (see (Bosq, 2000, Theorem 2.2, Page 46)), yields the result of Theorem 1. ■

Proofs of Theorems 4, 2 and 3

The proofs of Theorems 4 (i), 2 and 3 are respectively very similar to those of Theorems 1, 4 (ii) and 4 (iii). A brief outline of the proofs of the last two enumerations of Theorems 4 is given in the following. The proof of Theorem 4 (ii) is based on Theorem 4 (i) of this paper and Lemma 3 below in addition to the same arguments as those used in the proof of Theorem 3 in Cardot et al. (2003). Whereas the proof of Theorem 4 (iii) is obtained by the following Lemma 4 and Lemma 1, in addition to same the arguments as those used in the proof of Theorem 4 in Cardot et al. (2003). ■

Lemma 3. *Under Assumptions 1 and 2, we have*

$$\|\Gamma_{\mathbf{n}} - \Gamma\|_\infty = O_{a.s}\left(\frac{\log(\widehat{\mathbf{n}})}{\widehat{\mathbf{n}}}\right).$$

Proof. Recall that $\Gamma_{\mathbf{n}} = \widehat{\mathbf{n}}^{-1}\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} X_{\mathbf{i}} \otimes_H X_{\mathbf{i}}$ and $\Gamma = \mathbb{E}(X \otimes_H X)$. For all $\tau > 0$, we have

$$\begin{aligned}
\mathbb{P}\left(\|\Gamma_{\mathbf{n}} - \Gamma\|_\infty > \tau\right) &= \mathbb{P}\left(\sup_l \|\Gamma_{\mathbf{n}} v_l - \Gamma v_l\|_H > \tau\right) \\
&\le \sum_{l=1}^{+\infty} \mathbb{P}\left(\|\Gamma_{\mathbf{n}} v_l - \Gamma v_l\|_H > \tau\right) \\
&\le \frac{1}{\tau^2}\sum_{l=1}^{+\infty} \mathbb{E}\left(\|\Gamma_{\mathbf{n}} v_l - \Gamma v_l\|_H^2\right).
\end{aligned}$$

However, setting

$$L_{\mathbf{ij}} = \left\langle \langle X_{\mathbf{i}}, v_l\rangle_H X_{\mathbf{i}} - \mathbb{E}(\langle X, v_l\rangle_H X), \left\langle X_{\mathbf{j}}, v_l\right\rangle_H X_{\mathbf{j}} - \mathbb{E}(\langle X, v_l\rangle_H X)\right\rangle_H,$$

we have

$$\begin{aligned}
\mathbb{E}\left[\|\Gamma_{\mathbf{n}} v_l - \Gamma v_l\|_H^2\right] &= \mathbb{E}\left[\left\|\frac{1}{\widehat{\mathbf{n}}}\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}\left(\langle X_{\mathbf{i}}, v_l\rangle_H X_{\mathbf{i}} - \mathbb{E}(\langle X, v_l\rangle_H X)\right)\right\|_H^2\right] \\
&= \frac{1}{\widehat{\mathbf{n}}^2}\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} \mathbb{E}\left[\|\langle X_{\mathbf{i}}, v_l\rangle_H X_{\mathbf{i}} - \mathbb{E}(\langle X, v_l\rangle_H X)\|_H^2\right] + \frac{1}{\widehat{\mathbf{n}}^2}\sum_{\mathbf{i}\neq\mathbf{j}} \mathbb{E}\left(L_{\mathbf{ij}}\right) \\
&=: A + B.
\end{aligned}$$

Since $X_{\mathbf{i}}$ are strictly stationary with the same law as X, one has

$$\begin{aligned}
A &= \frac{1}{\widehat{\mathbf{n}}^2}\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}} \mathbb{E}\left[\|\langle X_{\mathbf{i}}, v_l\rangle_H X_{\mathbf{i}} - \mathbb{E}(\langle X, v_l\rangle_H X)\|_H^2\right] \\
&\le \frac{2}{\widehat{\mathbf{n}}^2}\sum_{\mathbf{i}\in\mathcal{I}_{\mathbf{n}}}\left(\mathbb{E}\left(\langle X_{\mathbf{i}}, v_l\rangle_H^2 \|X_{\mathbf{i}}\|_H^2\right) + \mathbb{E}\left(\langle X, v_l\rangle_H^2 \|X\|_H^2\right)\right) \\
&\le \frac{4M^2\lambda_l}{\widehat{\mathbf{n}}},
\end{aligned}$$

$$B = \frac{1}{\widehat{\mathbf{n}}^2}\sum_{0<\|\mathbf{i}-\mathbf{j}\|\le C_{\mathbf{n}}} \mathbb{E}(L_{\mathbf{ij}}) + \frac{1}{\widehat{\mathbf{n}}^2}\sum_{\|\mathbf{i}-\mathbf{j}\|>C_{\mathbf{n}}} \mathbb{E}(L_{\mathbf{ij}}) := B_1 + B_2,$$

where $0 < C_{\mathbf{n}} < \widehat{\mathbf{n}}$ and $C_{\mathbf{n}} \to \infty$ as $\mathbf{n} \to \infty$. However,

$$\mathbb{E}(L_{\mathbf{ij}}) = \sum_{s=1}^{+\infty} Cov\left(\langle X_{\mathbf{i}}, v_l\rangle_H \langle X_{\mathbf{i}}, v_s\rangle_H, \langle X_{\mathbf{j}}, v_l\rangle_H \langle X_{\mathbf{j}}, v_s\rangle_H\right)$$

and $\mathbb{E}\left(\langle X, v_l\rangle_H^4\right) \leq M^2 \mathbb{E}\left(\langle X, v_l\rangle_H^2\right) = M^2\lambda_l$. Then from Cauchy Schwartz inequality, we have

$$\begin{aligned}
|B_1| &\leq \frac{1}{\widehat{\mathbf{n}}^2} \sum_{k=1}^{C_{\mathbf{n}}} \sum_{\substack{\mathbf{i},\mathbf{j}\in\mathcal{I}_{\mathbf{n}} \\ k\leq\|\mathbf{i}-\mathbf{j}\|=t<k+1}} \sum_{s=1}^{+\infty} \mathbb{E}\left(\langle X_{\mathbf{i}}, v_l\rangle_H^2 \langle X_{\mathbf{i}}, v_s\rangle_H^2\right) \\
&\leq \frac{1}{\widehat{\mathbf{n}}^2} \sum_{k=1}^{C_{\mathbf{n}}} \sum_{\mathbf{j}\in\mathcal{I}_{\mathbf{n}}} \sum_{\substack{\mathbf{i}\in\mathcal{I}_{\mathbf{n}} \\ k\leq\|\mathbf{i}-\mathbf{j}\|=t<k+1}} \sum_{s=1}^{+\infty} \left[\mathbb{E}\left(\langle X_{\mathbf{i}}, v_l\rangle_H^4\right)\right]^{1/2} \left[\mathbb{E}\left(\langle X_{\mathbf{i}}, v_s\rangle_H^4\right)\right]^{1/2} \\
&\leq \frac{M^2\lambda_l^{1/2}}{\widehat{\mathbf{n}}} \sum_{k=0}^{C_{\mathbf{n}}} \sum_{\substack{\mathbf{i}\in\mathcal{I}_{\mathbf{n}} \\ k\leq\|\mathbf{i}\|=t<k+1}} \sum_{s=1}^{+\infty} \lambda_s^{1/2} \\
&\leq \frac{M^2\lambda_l^{1/2}}{\widehat{\mathbf{n}}} \sum_{t=1}^{C_{\mathbf{n}}} t^{d-1} \sum_{s=1}^{+\infty} \lambda_s^{1/2} \leq \frac{M^2\lambda_l^{1/2} C_{\mathbf{n}}^d}{\widehat{\mathbf{n}}} \sum_{s=1}^{+\infty} \lambda_s^{1/2}.
\end{aligned}$$

Taking $C_{\mathbf{n}} = \lfloor (\log(\widehat{\mathbf{n}}))^{1/d} \rfloor$ (where $\lfloor x \rfloor$ stands for the integer part of x), we obtain

$$|B_1| \leq \frac{c_2 \lambda_l^{1/2} \log(\widehat{\mathbf{n}})}{\widehat{\mathbf{n}}},$$

where c_2 is some positive constant. Besides, applying Lemma 2.1(i) of Tran (1990), we have

$$|\mathbb{E}(L_{ij})| \leq C \left\|\langle X_{\mathbf{i}}, v_l\rangle_H \langle X_{\mathbf{i}}, v_s\rangle_H\right\|_4 \left\|\langle X_{\mathbf{j}}, v_l\rangle_H \langle X_{\mathbf{j}}, v_s\rangle_H\right\|_4 \left[\alpha_{1,\infty}(|\mathbf{i}-\mathbf{j}|)\right]^{1/2}.$$

Since $\alpha_{1,\infty}(t) = O(t^{-\xi})$ with $\xi > 4d$, then

$$\begin{aligned}
|B_2| &\leq \frac{CM^3\lambda_l^{1/4}}{\widehat{\mathbf{n}}^2} \sum_{k=C_{\mathbf{n}}+1}^{\infty} \sum_{\substack{\mathbf{i},\mathbf{j}\in\mathcal{I}_{\mathbf{n}} \\ k\leq\|\mathbf{i}-\mathbf{j}\|=t<k+1}} \sum_{s=1}^{+\infty} \lambda_s^{1/4} \left[\alpha_{1,\infty}(t)\right]^{1/2} \\
&\leq \frac{CM^3\lambda_l^{1/4}}{\widehat{\mathbf{n}}} \sum_{k=C_{\mathbf{n}}+1}^{\infty} \sum_{\substack{\mathbf{i}\in\mathcal{I}_{\mathbf{n}} \\ k\leq\|\mathbf{i}\|=t<k+1}} \sum_{s=1}^{+\infty} \lambda_s^{1/4} \left[\alpha_{1,\infty}(t)\right]^{1/2} \\
&\leq \frac{CM^3\lambda_l^{1/4}}{\widehat{\mathbf{n}}} \sum_{s=1}^{+\infty} \lambda_s^{1/4} \sum_{t=1}^{\infty} t^{d-1} \left[\alpha_{1,\infty}(t)\right]^{1/2} \\
&\leq \frac{CM^3\lambda_l^{1/4}}{\widehat{\mathbf{n}}} \sum_{s=1}^{+\infty} \lambda_s^{1/4} \sum_{t=1}^{\infty} t^{d-1-\xi/2} \leq \frac{c_3\lambda_l^{1/4}}{\widehat{\mathbf{n}}}.
\end{aligned}$$

where c_3 is some positive constant. Therefore

$$\mathbb{E}\left(\|\Gamma_{\mathbf{n}} v_l - \Gamma v_l\|_H^2\right) \leq \frac{c_4 \lambda_l^{1/4} \log(\widehat{\mathbf{n}})}{\widehat{\mathbf{n}}}, \tag{14}$$

where c_4 is some positive constant. Hence $\mathbb{P}\left(\|\Gamma_{\mathbf{n}} - \Gamma\|_\infty > \tau\right) = O(\log(\widehat{\mathbf{n}})/\widehat{\mathbf{n}})$. ■

A rate of convergence of $\Gamma_{\mathbf{n}}$ with respect to norm $\|.\|_{L^2(\mathcal{HS})}$ appears in the following corollary.

Lemma 4. *Under Assumptions 1 and 2, we have*

$$\|\Gamma_{\mathbf{n}} - \Gamma\|_{L^2(\mathcal{HS})} = O\left(\sqrt{\frac{\log(\widehat{\mathbf{n}})}{\widehat{\mathbf{n}}}}\right).$$

Proof. By definition, we have

$$\|\Gamma_{\mathbf{n}} - \Gamma\|_{L^2(\mathcal{HS})} = \left\{\mathbb{E}\left[\|\Gamma_{\mathbf{n}} - \Gamma\|^2_{\mathcal{HS}}\right]\right\}^{1/2}.$$

Since $(v_j)_{j\geq 1}$ is an orthonormal basis of H, from the inequality in (14) we have

$$\mathbb{E}\left[\|\Gamma_{\mathbf{n}} - \Gamma\|^2_{\mathcal{HS}}\right] = \sum_{i=1}^{+\infty} \mathbb{E}\left[\|\Gamma_{\mathbf{n}}(v_i) - \Gamma(v_i)\|^2_H\right] \leq \frac{c_4 \log(\widehat{\mathbf{n}})}{\widehat{\mathbf{n}}} \sum_{i=1}^{+\infty} \lambda_i^{1/4}. \qquad \blacksquare$$

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