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Construction of Archimedean copulas using total time on test transforms

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In the present work, we propose a method of constructing Archimedean copulas using the total time on test transform, extensively used in reliability modelling. It is observed that the copula can be specified in terms of a univariate life distribution with a finite mean and monotone hazard rate. We discuss some new properties of the Kendall distribution arising from the proposed new generator and the associated measures of dependence.

Keywords: Ageing, Dependence measures, Excess wealth transform, Kendall distribution.

1. Introduction

Copulas have come to occupy a prominent role in statistical literature as a primary tool in modelling multivariate data by exploring the dependency structure between the constituent variables. Formally, a bivariate copula $C(u, v)$ is a function from $I^2 \to I$, $I = [0, 1]$ satisfying the properties:

(i) $C(u, 0) = C(0, v) = 0$,

(ii) $C(u, 1) = u$, $C(1, v) = v$,

(iii) for every u_1, u_2, v_1, v_2 in I such that $u_1 \leq u_2$, $v_1 \leq v_2$,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$

Thus, $C(u, v)$ satisfies the conditions for a bivariate distribution with uniform marginals on $[0, 1]$. More interestingly, if $H(x, y)$ is a bivariate distribution function with marginals F and G then there exists a copula

$$C(u, v) = H\left(F^{-1}(u), G^{-1}(v)\right),$$

where F^{-1} is the inverse of F and conversely, given a copula, we can find a bivariate distribution function

$$H(x, y) = C\left(F(x), G(y)\right),$$

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with marginals $F(x)$ and $G(y)$. Thus, from a given copula, we can generate a large class of bivariate distributions through different choices of marginals, but possessing the same dependence structure. Further, the dependence parameter can be estimated independently of the marginals. This flexibility and unique properties of a copula makes it an attractive choice in modelling problems. Currently, copulas find applications in quantitative finance in analysing risks, portfolio optimisation, analysing prices, among others, in civil engineering for analysing high bridge constructions, for simulation studies in earth quake engineering, mechanical and offshore constructions. It is also used in medicine for brain research, oncology and cardio vascular studies. Some other areas are hydrology, weather research, signal processing, reliability engineering, survival analysis and economics. We refer to Jouanin et al. (2007), Fan and Patton (2014), Burney et al. (2018), and Zhang and Singh (2019) and their references for more details.

A bivariate copula is said to be Archimedean if it can be written in the form

$$C(u, v) = \phi^{[-1]} \left(\phi(u) + \phi(v) \right),$$

where

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t), & 0 \le t \le \phi(0), \\ 0, & t > \phi(0), \end{cases}$$

and ϕ, called the generator, is a continuous, strictly decreasing, convex function from I to $(0, \infty)$ with $\phi(1) = 0$. Archimedean copulas (AC) are of special interest in view of (a) a large variety of useful copulas belong to this class, (b) its representation by a single function renders its easy construction, (c) provision to study scale free measures of dependence, and (d) ability to describe properties of bivariate distributions through some designated univariate distributions. These properties have encouraged a continuous stream of research to find new generators and new ACs with special properties. These include the use of Laplace transforms (Marshall and Olkin, 1988; Joe, 1997), composite functions (Frees and Valdez, 1998), lambda functions (Michiels et al., 2011), hyperbolic functions (Bal and Najjari, 2013), hyperbolic cotangent (Najjari et al., 2014), utility function (Spreeuw, 2014), distribution functions and probability generating functions (Alhadlaq and Alzaid, 2020), Lorenz curve (Fontanari et al., 2020), etc.

In this paper we present a new method of construction of ACs using the total time on test transform (TTT), an important concept in reliability theory as the generator. The definition and important properties of TTT are explained in the next section. The main motivation in choosing TTT as the generator is the expectation that many of the reliability aspects of TTT can be used to generate important properties of the resulting copulas. In Section 4, some results in this direction are established. For example, bivariate distributions serve as models of lifetimes of two-component devices. A crucial aspect to be observed in such cases is that the candidate model should match the type of dependence exhibited by the observed lifetimes. We demonstrate in the sequel that an ageing property such as increasing failure rate observed from the generator is equivalent to the negative dependence property of the bivariate life distribution. Similar results exist for other ageing behaviour of a univariate distribution associated with the generator. Thus, in our investigation, we have the double advantage of constructing a new AC as well as one satisfying our modelling requirements and satisfying other desirable properties discussed in the later sections.

The paper is organised into four sections. In Section 2, we review some of the existing definitions and results that are required for future deliberation. This is followed in Section 3, with the proposal of a new generator. Finally, Section 4 undertakes the discussion of some new properties of the Kendall distribution and dependence measures in the light of the generator based on TTT.

2. Preliminaries

Let X be a continuous nonnegative random variable with distribution function $F(x)$, survival function $\bar{F}(x)$ and quantile function

$$Q_X(u) = \inf\{x \mid F(x) \geq u\},\ 0 \leq u \leq 1.$$

When $F(x)$ is continuous and strictly increasing, $Q_X(u)$ becomes the ordinary inverse of $F(x)$. Assume that $\mu = E(X) < \infty$. The hazard rate and mean residual life functions of X are

$$h_X(x) = \frac{f(x)}{\bar{F}(x)}$$

and

$$m_X(x) = \frac{1}{\bar{F}(x)} \int_x^\infty \bar{F}(t)dt,$$

where f is the density function of X. In terms of the quantile function, we have the hazard quantile function and the mean residual quantile function as

$$h_Q(u) = h_X(Q_X(u)) = [(1-u)q_X(u)]^{-1}$$

and

$$m_Q(u) = m_X\left(Q_X(u)\right) = \frac{1}{1-u} \int_u^1 (1-p)q_X(p)dp,$$

where $q_X(u) = \frac{dQ_X(u)}{du}$ is the quantile density function of X.

An equipment or device is said to be ageing positively (negatively) if its remaining lifetime decreases (increases) as its age increases (decreases). Some important concepts in reliability theory that describe the nature of ageing are the following. We say that X is increasing (decreasing) hazard rate, IHR (DHR) if $h_X(x)$ $(h_Q(u))$ is increasing (decreasing) in x (u). Similarly X is increasing (decreasing) hazard rate average, IHRA (DHRA) if $\frac{1}{x}\int_0^x h(t)dt$ increasing (decreasing) in x and X is new better (worse) than used, NBU (NWU) if $\bar{F}(x, y) \leq (\geq) \bar{F}(x)\bar{F}(y)$. A detailed discussion of ageing concepts, their properties and applications can be found in Nair et al. (2013).

When several units are tested to ascertain their lifetimes, some of the units may fail within the prescribed time of the test, while others may survive. The sum of all completed and incomplete life lengths constitute the TTT. When the number of units are increased indefinitely, the TTT becomes

$$T(u) = \int_0^u (1-p)q_X(p)dp. \tag{1}$$

Some properties of $T(u)$ are:

(i) Whenever $F(x)$ is continuous, $T(u)$ is an increasing function of u with $T(0) = 0$ and $T(1) = \mu$. Also $T(u)$ is a quantile function with support $(0,\ \mu)$.

(ii) The distribution of X is uniquely determined by $T(u)$ as

$$Q_X(u) = \int_0^u \frac{T'(p)}{1-p} dp.$$

(iii) Also, $T'(u) = \left[h_Q(u)\right]^{-1}$ and $M_Q(u) = \frac{\mu - T(u)}{1-u}$. Further, X is IHR (DHR) if and only if $T(u)$ is concave (convex) in $[0,\ 1]$.

The above definitions and properties are available in Nair and Sankaran (2009) and Nair et al. (2013).

For the AC with generator ϕ, the function

$$K(t) = t - \frac{\phi(t)}{\phi'(t^+)}, \tag{2}$$

where prime denotes differentiation and $\phi(t^+)$ the limiting value of $\phi(x)$ as x tends to t from above, is the distribution function of the random variable $C(U, V)$, where U and V are uniform $(0, 1)$ random variables. We call (2) as the Kendall distribution function. See Nelsen (2006, Chapter 4) for detailed discussion of this distribution, and Susam and Ucer (2018, 2020) for additional references and applications to tests of hypotheses. A function $g(x)$ is said to be super (sub-) additive if $g(x + y) > (<) g(x) + g(y)$. Next, the definitions of certain dependence properties of Archimedean copulas are given: If C_ϕ is an Archimedean copula with generator ϕ then it is

(i) Positively (negatively) quadrant dependent, PQD (NQD) if and only if $-\log \phi^{-1}(t)$ is sub-additive (super-additive).

(ii) Stochastically increasing (SI) if and only if $\frac{-d}{dt}\phi^{-1}(t)$ is a log-convex function.

(iii) Left tail decreasing (LTD) if and only if $-\log \phi^{-1}(t)$ is a concave function.

We refer to Avérous and Dortet-Bernadet (2004) for further details of the above definitions.

3. Construction of an Archimedean copula

Continuing the notations in the previous section, we first obtain the conditions under which an Archimedean copula can be constructed using the TTT of X. Note that if the generator satisfies $\phi(0) = \infty$, we say that the AC is strict, otherwise it is non-strict.

Theorem 1. *Let X be a continuous nonnegative random variable with $E(X) < \infty$ and decreasing hazard rate. Then*

$$C_M(u,\ v) = M^{[-1]}\left(M(u) + M(v)\right), \tag{3}$$

is a non-strict Archimedean copula, where $M(u) = T(1-u)$, the mirror image of $T(u)$, the total time on test transform.

Proof. Since $T(0) = 0$, we have $M(1) = 0$. Also M is decreasing, since T is increasing. Being a decreasing hazard rate random variable, the TTT of X is convex and so is $M(u)$. Thus, M is the generating function of an Archimedean copula $C : I^2 \to I$ given by (3), which is not strict by virtue of $M(0) = T(1) = \mu < \infty$. ■

Example 1. Let X be a Pareto I random variable with quantile function

$$Q(u) = \sigma(1-u)^{-\frac{1}{\alpha}},\ 0 \leq u \leq 1,\ \alpha, \sigma > 0.$$

Then the TTT of X is

$$T(u) = \frac{\sigma}{\alpha - 1}\left((1-u)^{\frac{\alpha-1}{\alpha}} - 1\right),$$

giving

$$M(u) = \frac{\sigma}{\alpha - 1}\left(u^{\frac{\alpha-1}{\alpha}} - 1\right)$$

and

$$M^{-1}(u) = \left(1 + \frac{\alpha-1}{\sigma}u\right)^{\frac{\alpha}{\alpha-1}}.$$

Thus,

$$\begin{aligned} C_M(u,v) &= M^{-1}\left[\frac{\sigma}{\alpha-1}\left\{u^{\frac{\alpha-1}{\alpha}} - 1 + v^{\frac{\alpha-1}{\alpha}} - 1\right\}\right] \\ &= \max\left(u^{\frac{\alpha-1}{\alpha}} + v^{\frac{\alpha-1}{\alpha}} - 1, 0\right)^{\frac{\alpha}{\alpha-1}},\ \alpha \neq 1,\ \alpha \in [0,\ \infty], \end{aligned}$$

which is the Clayton copula.

For each generating function ϕ in the general definition

$$C_\phi(u,\ v) = \phi^{[-1]}\left(\phi(u) + \phi(v)\right), \tag{4}$$

Genest and Rivest (1993) observed that there exists a distribution function F_ϕ such that

$$F_\phi(u) = 1 - \phi^{-1}(u),\ 0 \leq u \leq 1, \tag{5}$$

and that when ϕ is continuous, convex, and decreasing, with $\phi(1) = 0$, F_ϕ is unimodal on $[0,\ \infty)$ with mode at zero. Later, Avérous and Dortet-Bernadet (2004) asserted that ϕ^{-1} is the survival function of the random variable $Z = 2\phi\left[\max\left(F_1(x),\ F_2(x_2)\right)\right]$, where F_1 and F_2 are marginal distribution functions of X_1 and X_2. For example, if $\phi(u) = -\log(u)$, the copula generated by $\phi(u)$ is $C_\phi = uv$, the independent copula denoted usually by Π. On the other hand, taking $M = -\log(u)$, we have $T(u) = -\log(1-u)$. The univariate distribution arising from ϕ is

$$F_\phi(x) = 1 - e^{-x},$$

the unit exponential distribution. At the same time, the Archimedean copula in (3) with M as above gives the quantile function

$$Q_T(u) = \int_0^u \frac{T'(p)}{1-p}dp = \frac{u}{1-u},$$

that corresponds with the Pareto II distribution with survival function $\bar{F}(x) = (1+x)^{-2},\ x > 0$. The distribution obtained from (5) and the baseline distribution from which the generator has been originated, are different.

It is also true that if ϕ and T correspond to the same distribution, then the copulas C_ϕ and C_M may be different. For example, the TTT that corresponds to the exponential distribution $\bar{F}(x) = \exp[-\lambda x]$ is $T(u) = \lambda^{-1}u$. Hence $M(u) = \lambda^{-1}(1-u)$ and

$$C_M(u,v) = M^{-1}\left(\frac{1-u}{\lambda}, \frac{1-v}{\lambda}\right) = \max(u+v-1, 0),$$

the Frechet-Hoeffding lower bound of every copula $C(u, v)$. But, $C_\phi(u, v) = uv$. Thus, it becomes evident that the properties of the copulas C_M and C_ϕ based on the quantile functions Q_M and Q_ϕ will be distinct, and therefore, the former is worth consideration. The TTTs corresponding Archimedean copulas, and their M functions of several distributions are exhibited in Table I.

The TTT, itself being a quantile function, possesses a TTT called the second order TTT. In this manner associated with a TTT, we can construct a hierarchy of TTTs with several interesting properties as discussed in Nair et al. (2008). For example, if the baseline distribution has increasing hazard rate, the higher order TTTs have lesser increasing hazard rates with eventually at some higher order, the hazard rate begins to decrease. Thus, by increasing the order of TTT one can reduce the hazard rate and thereby generate a new distribution with more reliable lifetime. While constructing ACs with such TTTs, it will be seen that we will be reducing the amount of negative dependence at each modification with eventually getting one with positive dependence. These results will be of practical importance in the choice of appropriate models for given data. The above method allows the construction of a hierarchy of Archimedean copulas by defining TTT of order n (Nair et al., 2008), defined as

$$T_n(u) = \int_0^u (1-p)t_{n-1}(p)dp, \quad n = 1, 2, \dots,$$

with $T_0(u) = Q(u)$, $t_n(u) = \frac{dT_n(u)}{du}$ and $\mu_{n-1} = \int_0^1 T_{n-1}(p)dp < \infty$, based on the fact that $T_1(u) = \int_0^u (1-p)q(p)dp$ is the quantile function in the support of $[0,\ \mu_0]$, $\mu_0 = E(X)$. One can take X_n to be the random variable with quantile function $T_n(u)$ defined on $[0,\ \mu_n]$. The distribution of X_n is specified by

$$t_n(u) = (1-u)t_{n-1}(u) = (1-u)^n t_0(u) = (1-u)^n q(u), \quad n \geq 1.$$

The Archimedean copula corresponding to X_n is

$$C_n(u, v) = M_n^{[-1]}\left(M_n(u) + M_n(v)\right),$$

where $M_n(u) = T_n(1-u)$.

Example 2. Let X be distributed as Pareto II with $Q(u) = \alpha\left[(1-u)^{-\frac{1}{c}} - 1\right]$, $\alpha,\ c > 0$. Then

$$\begin{aligned} T_n(u) &= \int_0^u (1-p)^n q(p)dp \\ &= \int_0^u (1-p)^n \frac{\alpha}{c}(1-p)^{-\frac{1}{c}-1}dp \\ &= \frac{\alpha}{cn-1}\left[1-(1-u)^{n-\frac{1}{c}}\right], \end{aligned}$$

giving

$$M(u) = \frac{\alpha}{cn-1}\left[1-u^{n-\frac{1}{c}}\right] \quad \text{and} \quad M^{-1}(u) = \left(1 - \frac{cn-1}{\alpha}u\right)^{\frac{c}{cn-1}}.$$

Then,

$$C_n(u, v) = \max\left[u^{\frac{cn-1}{c}} + v^{\frac{cn-1}{c}} - 1, 0\right]^{\frac{c}{cn-1}},\ c > \frac{1}{n-1}.$$

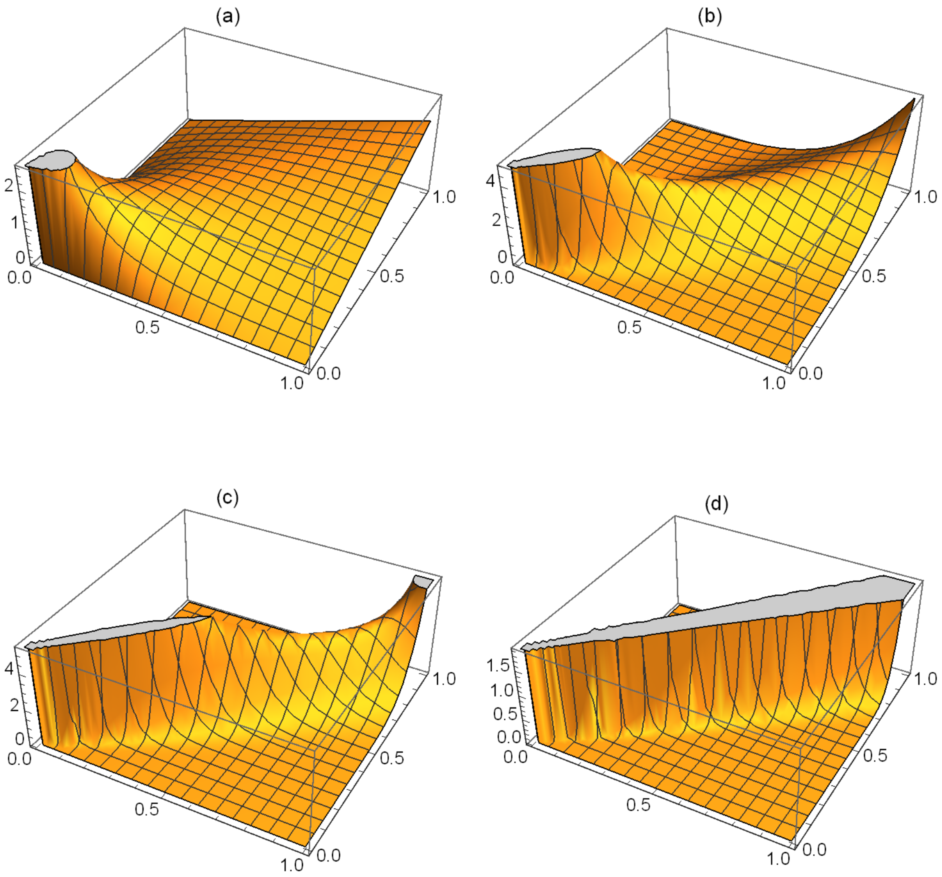

Figure 1. Density plots of Clayton copula with uniform marginals generated from the TTT of order (a) $n = 2$, (b) $n = 5$, (c) $n = 10$ and (d) $n = 20$ of the Pareto II distribution with parameter $c = 1$

For $n = 1$, $C_1(u,\ v)$ is the same as the Clayton copula in Example 1. When $n = 2,\ 3, \ldots$, we have a hierarchy of copulas as shown in Figure 1. Notice that how the change in the order of the generator can induce corresponding changes in shapes of the copulas in the figures.

Now, we consider the case when X has an increasing hazard rate. From Alsina et al. (2003), a function $A : I^2 \to I$ such that (i) $A(x,\ y) = A(y,\ x)$, (ii) $A(x,\ y) \leq A(z,\ w), x \leq z,\ y \leq w$, (iii) $A(A(x,\ y),\ z) = A(x,\ A(y,\ z))$, and (iv) $A(x, 1) = x$ for every x in I is called a T-norm, and each continuous T-norm is Archimedean if there exists an additive generator $t(x)$ which satisfies $A(x,\ y) = t^{-1}(t(x) + t(y))$. It becomes a copula if and only if $t(x)$ is convex. Otherwise it is an S-norm defined by $S : I^2 \to I$ satisfying (i), (ii), and (iii), and $S(x,\ 0) = x$. Also,

$$S(x,\ y) = 1 - A(1 - x,\ 1 - y).$$

Thus when X has an IHR, the TTT is concave and hence does not qualify to be a copula but only as an S-norm

$$C^*(u,\ v) = 1 - C(1-u,\ 1-v),$$

called the co-copula, which has a useful interpretation as

$$C^*(u,\ v) = P(U \le u\ \cup\ V \le v),$$

for uniform variates $U,\ V$ over $[0,1]$.

Since $T(1-u)$ cannot provide an Archimedean copula when X has IHR, we consider the alternative,

$$W(u) = \mu - T(u) = \int_u^1 (1-p)q(p)dp,$$

which is called the excess wealth transform of X. Some properties of $W(u)$, important references relative to it, and new applications are available in Nair and Vineshkumar (2021).

Theorem 2. *Let X be a continuous nonnegative random variable with finite expectation and increasing hazard rate. Then,*

$$C_W(u,\ v) = W^{[-1]}(W(u) + W(v)) \tag{6}$$

is a non-strict Archimedean copula.

Proof. We have $W(1) = 0$, $W(u)$ is decreasing, and $W'(u) = -(1-u)q(u) > 0$, so that $W(u)$ is convex. Thus, $W(u)$ is the generator of the Archimedean copula $C_W(u,\ v)$ and $W(0) = \mu < \infty$, shows that C_W is non-strict. ■

Example 3. Let X have rescaled beta distribution, which has an IHR with survival function

$$\bar{F}(x) = \left(1 - \frac{x}{R}\right)^c,\ 0 \le x \le R,\ c, R > 0,$$

and

$$Q(u) = R\left[1 - (1-u)^{\frac{1}{c}}\right].$$

This gives

$$W(u) = \frac{R}{c+1}(1-u)^{\frac{c+1}{c}}.$$

Using

$$W^{-1}(u) = 1 - \left[\frac{c+1}{R}u\right]^{\frac{c}{c+1}},$$

we write the corresponding Archimedean copula as

$$C_W(u,v) = \max\left[1 - \left\{(1-u)^{\frac{c+1}{c}} + (1-v)^{\frac{c+1}{c}}\right\}^{\frac{c}{c+1}},0\right],\ c \in [0,\ \infty].$$

More examples can be seen in Table 1.

Remark 1. The univariate distributions arising out of $\phi_1(u) = T(1-u)$ in Theorem 1 and $\phi_2(u) = W(u)$ in Theorem 2 are generally different. They respectively have the quantile functions

$$Q_{\phi_1}(u) = -\int_0^u \frac{\phi_1'(1-p)}{1-p}\,dp$$

and

$$Q_{\phi_2}(u) = -\int_0^u \frac{\phi_2'(-p)}{1-p}\,dp.$$

Remark 2. The two generators $T(1-u)$ and $W(u) = \mu - T(u)$ give the same copula if and only if X is exponential with $\bar{F}(x) = e^{-x}$. This is expected as the hazard rate being constant can be interpreted as either IHR or DHR. Moreover, the only solution of $T(1-u) = \mu - T(u)$ is $T(u) = u$, the TTT of the unit exponential law.

4. Kendall distribution

In this section, we discuss some new properties of the Kendall distribution, given in (2), of the copulas generated in the new framework.

Theorem 3. *The Kendall distribution $K(y)$ of the Archimedean copula generated by TTT and the distribution of X are uniquely determined by each other.*

Proof. When X has an IHR, the Kendall distribution is given by

$$K(y) = y - \frac{W(y)}{W'(y)}, \tag{7}$$

so that

$$\frac{d\log W(y)}{dy} = [y - K(y)]^{-1}.$$

This implies

$$\log W(y) = -\int_y^1 [p - K(p)]^{-1}dp$$

and

$$W(y) = \exp\left[-\int_y^1 (p - K(p))^{-1}dp\right].$$

Using the relationship between $W(u)$ and $q(u)$,

$$q(u) = \frac{-W'(u)}{1-u} = [(1-u)(u-K(u))]^{-1}\exp\left[-\int_u^1 (p-K(p))^{-1}dp\right],$$

the quantile density function of X, which determines the distribution of X. When X has a DHR, the proof is quite similar, with $M(u)$ replacing $W(u)$ and the definition of $T(u)$ in (1) to find $q(u)$. The converse follows from (7) and (1), and

$$q(u) = [(1-u)(u-K(u))]^{-1}\exp\left[\int_0^u (p-K(p))^{-1}dp\right]. \qquad \blacksquare$$

Table 1. Univariate distributions and Archimedean copulas.

Baseline survival function ($\bar{F}(x)$)	Baseline Quantile function ($Q(u)$)	Generator ($M(u)$ or $W(u)$)	Copula ($C_M(u,\ v)$ or $C_W(u,\ v)$)
$\left(1+\frac{x}{\alpha}\right)^{-c}$	$\alpha\left[(1-u)^{-\frac{1}{c}}-1\right]$	$M(u)=\frac{\alpha}{c}\left(1-u^{\frac{c-1}{c}}\right)$	$\left[\max\left(u^{-\theta}+v^{-\theta}-1,\ 0\right)\right]^{-\frac{1}{\theta}},\ \theta=\frac{1}{c}-1$
$\frac{\theta-1}{\theta-c^{\frac{1-\theta}{\theta}x}}$	$e^{\frac{\theta}{1-u}}\left(\frac{\theta+u-1}{\theta(1-u)}\right)-\frac{\theta-1}{\theta}e^{\theta}$	$M(u)=e^{\frac{\theta}{u}}-e^{\theta}$	$\theta\left[\log\left(e^{\frac{\theta}{u}}+e^{\frac{\theta}{v}}-e^{\theta}\right)\right]^{-1},\ \theta>0$
$\frac{(1-\theta)\exp\left(-\frac{1-\theta}{\theta}x\right)}{1-\theta\exp\left(-\frac{1-\theta}{\theta}x\right)}$	$\frac{\theta}{1-\theta}\log\frac{1-\theta u}{1-u}$	$M(u)=-\log(\theta u+1-\theta)$	$\max\left[\theta uv+(1-\theta)(u+v-1),\ 0)\right],\ 0\leq\theta\leq 1$
$\frac{2}{1+e^{\frac{x}{\sigma}}}$	$\sigma\log\frac{1+u}{1-u}$	$W(u)=2\sigma\log\frac{2}{1+u}$	$\max\left[\frac{u+v+uv-1}{2},\ 0\right]$
$\frac{2+(\alpha-1)e^{(1+\alpha)x}}{1+\alpha e^{(1+\alpha)x}}$	$\frac{1}{1+a}\log\frac{a+u}{a(1-u)}$	$W(u)=\log\frac{a+1}{a+u}$	$\frac{(a+u)(a+v)}{a+1}-a,\ a>0$

Example 4. Consider the generator $W(u) = (1-u)^{\theta}$, $\theta > 1$. Then $q(u) = \theta(1-u)^{\theta}$ and $Q(u) = \left(1 - \frac{\theta-1}{\theta}u\right)^{\frac{1}{\theta-1}}$, giving the distribution of X as

$$\bar{F}(x) = \left(1 - \frac{x}{R}\right)^{c}, \quad 0 \le x \le R, \quad c > 0,$$

where $c = \frac{1}{\theta-1}$ and $R = \frac{\theta}{\theta-1}$. Also, $K(u) = \frac{1}{\theta}(1 + \theta u - u)$, $\frac{-1}{\theta-1} \le u \le 1$, showing that $K(u)$ is the uniform distribution function over $\left(\frac{-1}{\theta-1}, 1\right)$. More examples are given in Table 2.

There are several properties of the AC constructed by the new method and the Kendall distribution in reliability theory, as well as in the selection of copulas on the basis of dependence criteria. These are discussed in the rest of this section. The basic concepts in modelling bivariate data using copulas and their applications can be found in Nair et al. (2018).

In the case of IHR random variables,

$$W(u) = \int_{u}^{1} (1-p)q(p)dp = (1-u)m_Q(u),$$

and

$$W'(u) = -(1-u)q(u) = -h_Q^{-1}(u).$$

Similarly, when X has a DHR, $T(u) = \mu - M(u)$ and $T'(u) = 1/h_Q(u)$. Thus, we have the Kendall distribution expressed in terms of the hazard and mean residual quantile functions.

Theorem 4. *The Kendall distribution corresponding to X is given by*

$$K(u) = \begin{cases} u + (1-u)m_Q(u)h_Q(u) & \text{when } X \text{ has an IHR}, \\ u + \left[\mu - u m_Q(1-u)\right] h_Q(1-u) & \text{when } X \text{ has a DHR}. \end{cases}$$

Example 5. Assume that X is distributed as half-logistic with

$$\bar{F}(x) = 2\left[1 + \exp\left(\frac{x}{\sigma}\right)\right]^{-1},$$

or equivalently

$$Q(u) = \sigma \log\left(\frac{1+u}{1-u}\right), \quad \sigma > 0.$$

We have $h_Q(u) = (2\sigma)^{-1}(1+u)$ and $m_Q(u) = \frac{2\sigma}{1-u} \log\left(\frac{2}{1+u}\right)$, so that

$$K(u) = u + (1+u)\log\left(\frac{2}{1+u}\right).$$

Since most of the life distributions have known forms for their $h_Q(u)$ and $m_Q(u)$, it is easier to calculate $K(u)$ directly from them, as seen from Table 3.

In modelling problems, the most important aspect of a copula is the nature of dependency it offers. Several approaches to measure the dependency among the variables in a copula are available in the literature. One of these is to evaluate the extent of the dependence through some global measures of

Table 2. Baseline distributions and Kendall distributions.

$q(u)$	$M(u)$ or $W(u)$	$K(u)$	Copula
$\frac{\theta(-\log u)^{\theta-1}}{u(1-u)}$	$W(u) = (-\log u)^{\theta}$	$u - \frac{u \log u}{\theta}$	$\exp\left[-\left((-\log u)^{\theta} + (-\log v)^{\theta}\right)\right]^{1/\theta}$
$\frac{e^{\theta/u}}{(1-u)u^2}$	$W(u) = e^{\theta/u} - e^{\theta}$	$u + u^2\left(1 - \theta e^{-\theta/u}\right)$	$\theta\left[\log\left(e^{\theta/u} + e^{\theta/v} - e^{\theta}\right)\right]^{-1}$
$(1-u)^{-(\theta+2)}$	$M(u) = \frac{1}{\theta}(u^{-\theta} - 1)$	$u - \frac{u}{\theta^2}(1 - u^{\theta})$	$\max\left[u^{-\theta} + v^{-\theta} - 1,\ 0\right]^{-1/\theta}$
$\frac{\theta}{(\theta(1-u)+1-\theta)(1-u)}$	$M(u) = -\log(\theta u + 1 - \theta)$	$u + \frac{1}{\theta}(\log(\theta u + 1 - \theta))(\theta u + 1 - \theta)$	$\theta uv + (1-\theta)(u + v - 1)$
$\frac{\theta}{(1-u)^2[1-\theta\log(1-p)]}$	$M(u) = \log(1 - \theta \log u)$	$u + \frac{\log(1-\theta\log u)(1-\theta\log u)}{\theta}$	$uv \exp\left[-\theta \log u \log v\right]$
$\frac{\theta u^{\theta-1}}{(1-u)^{\theta+2}}$	$M(u) = \left(\frac{1}{u} - 1\right)^{\theta}$	$\frac{1}{\theta}u(\theta - 1 + u)$	$\left[1 + \left((u^{-1} - \theta)^{\theta} + (v^{-1} - \theta)^{\theta}\right)^{1/\theta}\right]^{-1}$

Table 3. Life distributions and their Kendall versions.

Distribution	$Q(u)$	$h_Q(u)$	$m_Q(u)$	$K(u)$
Rescaled beta	$R(1-(1-u))^{\frac{1}{c}},\ c,\ R>0$	$\frac{c}{R}(1-u)^{-\frac{1}{c}}$	$\frac{R}{c+1}(1-u)^{\frac{1}{c}}$	$u+\frac{c}{c+1}(1-u)$
Power	$\alpha u^{\frac{1}{\beta}},\ \alpha,\ \beta>0$	$\frac{\beta}{\alpha}\frac{u^{1-1/\beta}}{1-u}$	$\frac{\beta}{1-u}\left[u^{1-1/\beta}-\frac{u^{1-(1/\beta)+1}}{\beta+1}\right]$	$u+\frac{\beta}{\beta+1}\frac{u^{1-1/\beta}}{1-u}\left(\beta-u^{1/\beta}(\beta+1-u)\right),\ \beta\geq 1$
Pareto II	$\alpha\left((1-u)^{-1/c}-1\right),\ \alpha,\ c>0$	$\frac{c}{\alpha}(1-u)^{1/c}$	$\frac{\alpha}{c-1}(1-u)^{-1/c}$	$u+\frac{cu}{c-1}\left(u^{1/c-1}-1\right)$
Exponential geometric	$\frac{1}{\lambda}\log\frac{1-au}{1-u},\ \lambda>,\ 0<a<1$	$\frac{\lambda}{1-a}(1-au)$	$\frac{1-a}{\lambda a(1-u)}\log\frac{1-au}{1-a}$	$u-\frac{1}{a}(1-a(1-u))\log(1-a(1-u))$
Generalised Pareto	$fracba\left((1-u)^{-a/(a+1)}-1\right),\ a>0,\ b>-1$	$\frac{a+1}{b}(1-u)^{a/(a+1)}$	$b(1-u)^{-a/(a+1)}$	$a+1-au, a<0$
				$au\left[\frac{a+1}{u}(1-u)^{-1/(a+1)}-1\right],\ a>0$
Linear mean residual quantile function	$-(c+\mu)\log(1-u)-2c\mu,\ c>0,\ -c<\mu<c$	$(\mu-c+2cu)^{-1}$	$\mu+cu$	$u+\frac{(1-u)(\mu+cu)}{\mu-c+2cu}, c<0$

association. The Pearson coefficient of correlation, Kendall's tau, Spearman's rho, and Blomquest's beta are some of the important measures in this connection. Of these, Kendall's tau has a simple form for the Archimedean copula. For an Archimedean copula with generator ϕ, Kendall's tau has the expression

$$\tau = 1 + 4\int_0^1 \frac{\phi(u)}{\phi'(u)}du.$$

When TTT is used to find the generator, the Kendall coefficient becomes

$$\tau = \begin{cases} 1 + 4\int_0^1 \frac{W(u)}{W'(u)}du, & \text{when } X \text{ has an IHR,} \\ 1 + 4\int_0^1 \frac{M(u)}{M'(u)}du, & \text{when } X \text{ has a DHR.} \end{cases}$$

Making use of the reliability functions

$$\tau = \begin{cases} 1 - 4\int_0^1 (1-u)m_Q(u)h_Q(u)du, & \text{when } X \text{ has an IHR ,} \\ 1 - 4\int_0^1 \left[\mu - m_Q(u)\right] h_Q(u)du, & \text{when } X \text{ has a DHR .} \end{cases}$$

An advantage of the above formula is that the knowledge of the copula or generator is not essential in calculating the dependence measure of the associated copula. This aspect becomes quite useful in the choice of the copula appropriate to given data on the basis of the value of τ.

Example 6. Consider the mean residual quantile distribution (Midhu et al., 2013)

$$Q(u) = -(c+\mu)\log(1-u) - 2cu, \quad -\mu \le c \le \mu, \quad \mu > 0.$$

For this distribution, $m_Q(u) = \mu + cu$ and $h_Q(u) = (\mu - c + 2cu)^{-1}$. Assume that $c < 0$ so that $h_Q(u)$ is increasing. Then the Kendall's tau is

$$\tau = 1 - 4\int_0^1 \frac{(1-u)(\mu+cu)}{\mu - c + 2cu}du = 2(\mu - c) + 4\left(c + \frac{(c-\mu)^2}{2}\right)\log\frac{\mu - c}{\mu + c}.$$

A second approach to verify dependence is to define certain properties and classify bivariate distributions accordingly. The properties like PQD, SI, LTD, and LCSD defined in Section 2 belong to this category. For Archimedean copulas the following results are true.

Theorem 5. *An Archimedean copula is*

(i) PQD (NQD) if and only if $-\log W^{-1}(u)$ *is sub-additive (super-additive) when X is IHR and* $-\log M^{-1}(u)$ *is sub-additive (super-additive) when X has a DHR.*

(ii) SI(Y|X) (SI(X|Y)) if and only if $-dW^{-1}(u)/du$ $(-dM^{-1}(u)/du)$ *is log-convex when X has an IHR (DHR).*

(iii) LTD if $-\log W^{-1}(u)$ $(-\log M^{-1}(u))$ *is concave when X has an IHR (DHR).*

Regarding tail dependence, the relevant results are stated in the next theorem.

Theorem 6. *If C is an Archimedean copula, then the upper and lower tail dependence parameters are*

$$\lambda_U = \begin{cases} 2 - \lim_{x\to 0} \frac{1-M^{-1}(2x)}{1-M^{-1}(x)}, & \text{if } X \text{ has a DHR,} \\ 2 - \lim_{x\to 0} \frac{1-W^{-1}(2x)}{1-W^{-1}(x)}, & \text{if } X \text{ has an IHR,} \end{cases}$$

and

$$\lambda_L = \begin{cases} \lim_{x\to \infty} \frac{M^{-1}(2x)}{M^{-1}(x)}, & \text{if } X \text{ has a DHR,} \\ \lim_{x\to \infty} \frac{W^{-1}(2x)}{W^{-1}(x)}, & \text{if } X \text{ has an IHR.} \end{cases}$$

Another important result is that the type of dependence in the AC can be determined from the $T(u)$ function. Since $T(u)$ is a quantile function, $T(1-u)$ represents the corresponding survival function of a random variable, say S. One can deduce from Avérous and Dortet-Bernadet (2004), Propositions 1 and 5 that

(i) S is NBU (NWU) if and only if C is NQD (PQD), and

(ii) S is IHR (DHR) if and only if C is LTI (LTD).

To conclude this work, it is observed that Archimedean copulas can be generated in terms of TTT of a nonnegative random variable, with finite mean and either increasing or decreasing hazard rate. These generators are different from the conventional ones and therefore gives scope for using them in the place of conventional generators. We have spotted some applications of the results. The time-dependent measures of association and stochastic ordering of Archimedean copulas can also be derived from the properties of the life distribution. These aspects are being investigated and will be reported elsewhere.

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