ROBUST AND EFFICIENT ESTIMATION OF THE SHAPE PARAMETER OF ALPHA-STABLE DISTRIBUTIONS

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Summary: In this paper we consider robust and efficient estimators of the shape parameter of symmetric alpha-stable distributions obtained by using the Minimum Density Power Divergence method introduced in Basu, Harris, Hjort and Jones (1998). We established their high asymptotic efficiency and verified these results in simulations. The functionals corresponding to the estimators have bounded influence functions and simulations confirm their robustness when the sample distribution is in a vicinity of the model distribution. The simulations also show that the Minimum Density Power Divergence Estimators (MDPDEs) of the shape parameter of alpha-stable distributions have superior performance over other existing estimators. The high efficiency combined with robustness of the MDPDEs in estimating the shape parameter of alpha-stable distributions make them an attractive alternative to the preceding estimation procedures considered in the literature.

1. Introduction

Alpha-stable distributions were discovered by Paul Lévy (cf. Lévy (1925)) and are characterized by four parameters: $\alpha \in (0,2]$, $\beta \in [-1,1]$, $\sigma > 0$ and $\mu \in R$ which have interpretations of shape, skewness, scale and location, respectively. They have found numerous practical applications in a number of Sciences ranging from Physics, Biology, Geology, Ecology to Finance. Due to their attractive theoretical properties and rapidly growing access to computing power these distributions have enjoyed an increasing popularity over the last three decades. Nonetheless, working with alpha-stable models in practice remains even nowadays a non-trivial task. The density functions of these distributions have no closed form representation except the cases of Gaussian ($\alpha = 2$), Cauchy ($\alpha = 1$) and Lévy ($\alpha = \frac{1}{2}$) distributions, respectively.

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If $0 < \alpha < 2$ the alpha-stable distributions have finite p-th order moments for $0 < p < \alpha$ but infinite p-moments if $p \geq \alpha$. Since the alpha-stable distribution with $\alpha = 2$ is gaussian, it has all p-th order moments finite. Consequently, if $\alpha < 2$ the infinite variance of alpha-stable random variables implies that rare observations of very large magnitude can be expected, which may dominate the sum of all other remaining random variables. Thus, it is not advisable to treat these observations as outliers since excluding them from the sample results in a loss of information available in the original data. In many cases these observations may be even of the greatest interest, cf. Huber and Ronchetti (2009) and Hampel, Ronchetti, Rousseeuw and Stahel (1986).

The problem of parameter estimation for alpha-stable random variables has been studied extensively. The earliest estimation methods were developed by Fama and Roll (1971). Their approach was based on empirical quantiles. They obtained estimates of the shape parameter α for $\alpha > 1$ but omitted the case when $\alpha \leq 1$. A year later Press (1972) developed a method for estimating parameters of alpha-stable distributions using empirical characteristic functions. By using the properties of the plot of the log-log characteristic function, Koutrouvelis (1980) developed a method for estimating the shape parameters based on linear regression. A further improvement of this method was given in Koutrouvelis (1981), yet even the improved estimator suffers of high bias (see Weron (2001)). Other authors who also have used the sample characteristic function include Paulson, Holcomb and Leitch (1975), Feuerverger and McDunnough (1981) and Kogon and Williams (1998). McCulloch (1986) developed a method based on quantiles to estimate the shape parameter for $\alpha \geq 0.6$. However, as he noted, his method performed very poorly when $\alpha \leq 0.6$. Badahdah and Siddiqui (1991) have also used empirical quantiles as well as other statistics such as the trimmed and the Winsorised means to estimate parameters of these distributions. DuMouchel (1973) made the first attempt to develop the maximum likelihood estimation in the case of alpha-stable distributions. He pointed out that the maximum likelihood method was not robust. Nolan (1997) and Nolan (2001) further explored and developed the maximum likelihood estimation method for the parameters of alpha-stable distributions.

To obtain robust and efficient estimators of the shape parameter α of symmetric stable distributions, we apply an estimation procedure introduced in Basu et al. (1998). In the present case it minimizes a divergence between two densities: the density function of a symmetric alpha-stable distribution and the true, though unknown, probability density function.

The paper is organized as follows. In Section 2 we recall M-parametrization and basic features of alpha-stable distributions which will be needed in the sequel. In Section 3 we consider the Minimum Density Power Divergence Estimators (MDPDEs) as estimators of the shape parameter α of stable distributions. In Sections 4 and 5 we compute the influence functions and the asymptotic relative efficiencies of these estimators. In Section 6 we report our simulations for both uncontaminated and contaminated models. Finally, in Section 7 we present our conclusions.

2. M-parametrization and numerical problems of alpha-stable distributions

A detailed discussion of the problems related to the probability density functions of alpha-stable distributions can be found in Zolotarev (1964), Zolotarev (1986), Janicki and Weron (1994), Samorodnitsky and Taqqu (1994) and Uchaikin and Zolotarev (1999). We shall use Zolotarev's M-parametrization of stable distributions recalled in Definition 1 below. This parametrization ensures that the probability density functions and probability distribution functions are jointly continuous in parameters, cf. Cheng and Liu (1997), Nolan (1997) and Zolotarev (1986).

Definition 1 A random variable Y is stable if and only if $Y = aZ + b$, where $a > 0$, $b \in \Re$ and Z is a random variable with a characteristic function

$$E(\exp(itZ)) = \begin{cases} \exp\left\{-\sigma^\alpha |t|^\alpha (1 - i\beta sign(t)\tan(\frac{\pi\alpha}{2})(|\sigma t|^{1-\alpha} - 1)) + i\mu t\right\}, & \text{if } \alpha \neq 1, \\ \exp\left\{-\sigma|t|(1 + i\beta sign(t)\frac{2}{\pi}\ln(\sigma|t|)) + i\mu t\right\}, & \text{if } \quad \alpha = 1. \end{cases} \tag{1}$$

In formula (1) parameter $0 < \alpha \leq 2$ determines the tail weight and is called the shape parameter, the skewness parameter $\beta \in [-1, 1]$ determines the asymmetry of the distribution, $\sigma > 0$ is the scale parameter and determines the spread of the distribution, and $\mu \in R$ is the location parameter and determines the location of the mode of the distribution.

As we already stated in Section 1, with the exception of the three special cases (Gaussian, Cauchy and Lévy distributions), the probability density functions of alpha-stable distributions do not have closed form expressions. However it is known that they exist and have bounded continuous derivatives of all orders on their support (Zolotarev (1986), p. 23).

We focus our attention on estimating the shape parameter α and assume that the other parameters are known. Hence, without loss of generality in our case we can assume for simplicity that $a = 1, b = 0, \beta = 0, \sigma = 1$ and $\mu = 0$. Then the characteristic function of Y reduces to the following form

$$E(\exp(itY)) = \exp\{-|t|^\alpha\}, \alpha \in (0, 2]. \tag{2}$$

It will be convenient to use the following notation for the random variable Y in formula (2)

- $F(y; \alpha)$ denotes a cumulative distribution function (cdf) of Y,
- $f(y; \alpha)$ denotes a probability density function (pdf) of Y and
- S_α denotes the probability distribution of Y.

Although alpha-stable distributions have characteristic functions of a closed form, it is still hard to calculate their densities by direct application of the inversion theorem because it involves integrals of highly oscillating functions over unbounded regions. Several methods have been proposed in the literature to numerically compute density functions of alpha-stable distributions. DuMouchel (1973) and Holt and Crow (1973) tabulated densities for alpha-stable distributions for selected values of α and skewness parameter β. We will use Zolotarev's integral representations of the densities, cf. Zolotarev (1986), Chapter 2. Nolan (1997) discussed accurate algorithms for computing general alpha-stable densities with $\alpha > 0.1$ using this Zolotarev's M-parameterizations. More recently, Takemura and Matsui (2006) combined the Nolan's approach and asymptotic series expansions to compute the densities of symmetric alpha-stable distributions. Other asymptotic series expansions that have been used in the literature include Bergström (1952), Section XVII.6 of Feller (1966), Chapter 2 of Ibragimov and Linnik (1971), and Chapter 5 of Lukacs (1970). In our work it was necessary to improve and control the precision of the method reported in Takemura and Matsui (2006). For the sake of brevity of presentation we omit here the technical details for which we refer to Chapter 2 of Kangogo (2012).

3. The MDPDEs for symmetric alpha-stable distributions

Our estimation method of the shape parameter α is based on the minimum density power divergence (MDPD) introduced in Basu et al. (1998). Details of the asymptotic properties of the MDPDEs are also discussed in the general case in Juárez and Schucany (2006).

We consider the parametric family of symmetric alpha-stable distributions $F(y;\alpha)$ whose probability density functions are denoted by $f(y;\alpha)$. Let $\gamma \in (0,1]$ denote a parameter of the MDPD method that controls a trade-off between robustness and efficiency of the estimators of α.

The divergence between $f(y;\alpha)$ and the true unknown density function $f(y)$ is given by

$$d_\gamma(f(y;\alpha),f) = \int \left\{ f^{1+\gamma}(y;\alpha) - (1+\frac{1}{\gamma}) f(y) f^{\gamma}(y;\alpha) + \frac{1}{\gamma} f^{1+\gamma}(y) \right\} dy.$$

The MDPD functional chooses α such that the density $f(y;\alpha)$ is as close as possible to the true unknown p.d.f. $f(y)$ of the population. To see that this functional can be considered as a function of the true cdf F we present it in an equivalent way in the following form

$$\alpha_\gamma(F) = \arg\min_\alpha \left[\int f^{1+\gamma}(y;\alpha)dy - (1+\frac{1}{\gamma}) E_F f^{\gamma}(Y;\alpha) + \frac{1}{\gamma} E_F(f^{\gamma}(y)) \right], \tag{3}$$

where Y is a random variable with the distribution function $F(y)$. The third term of the right-hand side of equation (3) does not depend on α. Therefore, minimising expression (3) with respect to α is equivalent to the following minimisation problem

$$\alpha_\gamma(F) = \arg\min_\alpha \left[\int f^{1+\gamma}(y;\alpha)dy - (1+\frac{1}{\gamma}) E_F f^{\gamma}(Y;\alpha) \right]. \tag{4}$$

Since $f(y;\alpha)$ is a bounded and continuous probability density function, cf. Theorem 5.8.1 of Lukacs (1970), the integral and the expected values in equation (4) are finite.

Let $\hat{F}_n(y)$ denote the empirical distribution function of a sample $Y_1, Y_2, \ldots, Y_n$, with Y_i's having cdf F. By evaluating the functional α_γ given by formula (4) at $\hat{F}_n$, we get the MDPD estimator of α

$$\alpha_\gamma(\hat{F}_n) = \arg\min_\alpha \left[\int f^{1+\gamma}(y;\alpha)dy - (1+\frac{1}{\gamma}) \frac{1}{n} \sum_{j=1}^{n} f^{\gamma}(Y_j;\alpha) \right].$$

The estimator

$$\hat{\alpha}_{\gamma,n} = \alpha_\gamma(\hat{F}_n)$$

belongs to the class of general M-estimators introduced in Huber (1967) and can be obtained as a solution of the equation

$$\frac{d}{d\alpha} \left[\int f^{1+\gamma}(y;\alpha)dy - (1+\frac{1}{\gamma}) \frac{1}{n} \sum_{j=1}^{n} f^{\gamma}(Y_j;\alpha) \right] = 0.$$

Passing with differentiation under the integral sign, we obtain the estimating equation

$$\int f^{\gamma}(y;\alpha) \frac{\partial}{\partial\alpha} f(y;\alpha)dy - \frac{1}{n} \sum_{j=1}^{n} f^{\gamma-1}(Y_j;\alpha) \frac{\partial}{\partial\alpha} f(Y_j;\alpha) = 0 \tag{5}$$

which is effectively used to find numerically $\hat{\alpha}_{\gamma,n}$. We applied here an efficient and robust MM-algorithm for solving equations introduced in Kozek and Trzmielak-Stanisławska (1988) and Kozek and Trzmielak-Stanisławska (1989) and available from Matlab Central at
`http://www.mathworks.com/matlabcentral/fileexchange/29253-rootktsmm`.

4. Influence functions of MDPDEs at the symmetric alpha-stable distributions

For the sake of clarity of presentation of our results we present below, in Proposition 1 and Corollary 1, the form of the influence function for the functional $\alpha_\gamma(F)$ given by formula (4) or, equivalently, by equation (5). Proposition 1 is an immediate consequences of Theorem 2 of Basu et al. (1998) and of their formula for influence function given in section 3.3. Related results can also be found in Theorem 5 of Juárez and Schucany (2006). We need these results to prove Theorem 1 implying robustness of the MDPDEs.

Proposition 1 (Basu et al. (1998), Section 3) Let $Y_1, \cdots, Y_n$ be i.i.d random variables with a cumulative distribution function F and a probability density function f. If $f(y;\alpha)$ is a probability density function of a symmetric alpha-stable distribution and $\gamma \in (0,1]$ is a parameter of the MDPD method then the expression for the influence function of $\alpha_\gamma(F)$ is given by

$$IF(x;\alpha_\gamma(F),F) = J_{\gamma,F}^{-1}\left\{u(x;\alpha)f^\gamma(x;\alpha) - \zeta_F\right\} \tag{6}$$

where

$$J_{\gamma,F} = \int u^2(y;\alpha)f^{1+\gamma}(y;\alpha)dy + \int \left\{i(y;\alpha) - \gamma u(y;\alpha)^2\right\}\left\{f(y) - f^\gamma(y;\alpha)\right\} f^\gamma(y;\alpha)dy, \tag{7}$$

$$\zeta_F = \int u(y;\alpha)f^\gamma(y;\alpha)f(y)dy \tag{8}$$

and where $u(y;\alpha) = \partial \log f(y;\alpha)/\partial\alpha$ and $i(y;\alpha) = -\partial u(y;\alpha)/\partial\alpha$.

Corollary 1 If we take $f = f(.;\alpha)$ and note that

$$u(y;\alpha) = f^{-1}(y;\alpha)\frac{\partial}{\partial\alpha}f(y;\alpha),$$

then formula (6) simplifies to

$$IF(x;\alpha_\gamma(F_\alpha),F_\alpha) = J_{\gamma,F_\alpha}^{-1}\Psi(x;\alpha_\gamma,F_\alpha) \tag{9}$$

where

$$J_{\gamma,F_\alpha} = \int f^{\gamma-1}(y;\alpha)\left(\frac{\partial}{\partial\alpha}f(y;\alpha)\right)^2 dy \tag{10}$$

and

$$\Psi(x;\alpha_\gamma(F_\alpha),F_\alpha) = f^{\gamma-1}(x;\alpha)\frac{\partial}{\partial\alpha}f(x;\alpha) - \int f^\gamma(y;\alpha)\frac{\partial}{\partial\alpha}f(y;\alpha)dy. \tag{11}$$

Recall that the density function of a symmetric alpha-stable distribution $f(.;\alpha)$ is unimodal, bounded and has uniformly bounded derivatives of all orders, cf. Zolotarev (1986), Lukacs (1970), Takemura and Matsui (2006). Hence we infer from formula (11) that $\Psi(x;\alpha_\gamma(F_\alpha),F_\alpha)$ is bounded as a function of x and that it converges to 0 when either $x \to \infty$ or $x \to -\infty$. It is clear that for alpha-stable distributions J_{γ,F_α} given by formula (10) is positive. So, we have the following Theorem.

Theorem 1 For alpha-stable distributions the influence function $IF(x;\alpha_\gamma(F_\alpha),F_\alpha)$ given by formula (9) is bounded as a function of x and

$$\lim_{|x|\to\infty} IF(x;\alpha_\gamma(F_\alpha),F_\alpha) = 0.$$

Let us emphasize that Theorem 1 implies robustness of the of the MDPDEs. Bounded influence curve implies finite gross error sensitivity of the MDPDEs and the fact that the influence curve converges to zero when $|x| \to \infty$ implies that sensitivity of MDPDEs to large observations vanishes with their size increasing. We refer to Huber (1964) and Hampel et al. (1986) for comprehensive discussion of the role and interpretation of the influence function and for a comprehensive theory of Robust Statistics.

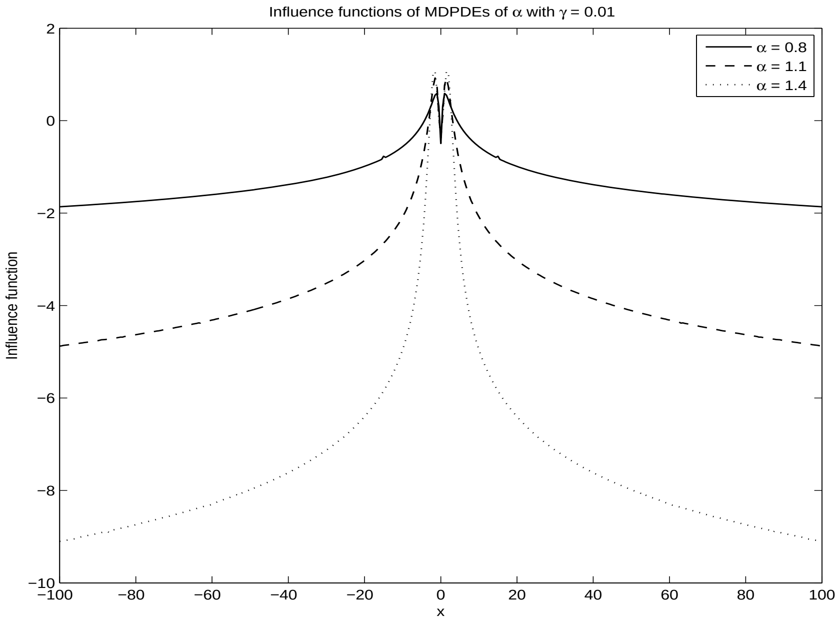
By formula (9) it becomes evident that computation of the influence function requires in the present case evaluation of the alpha-stable probability density function, it's derivative with respect to the shape parameter α and a numerical evaluation of integrals over unbounded region R.

The difficulty in guaranteeing in such cases a reliable integration is related to heavy-tails of the alpha-stable distributions. Restricting in a numerical integration support of the integrands to a fixed region like $[-10^k, 10^k]$ with small values of k is not acceptable because in many cases, the contributions coming from an exterior of such a support have non-negligible values.

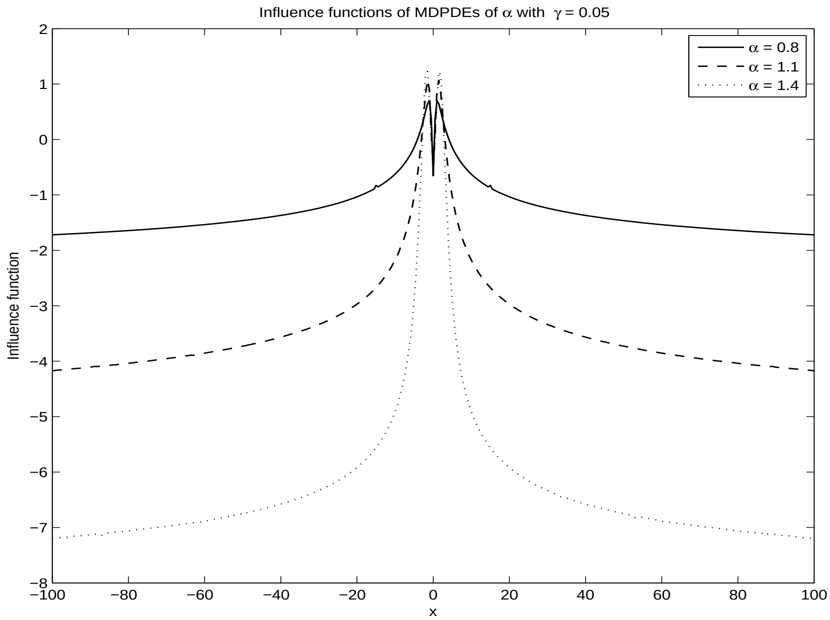
On the other hand, it is not easy to carry out precise numerical integration over a large region for integrands with variation changing over the support. Our integrands are quickly varying in a vicinity of zero and then they are decaying to zero slowly, often extremely slowly. Hence, it is necessary to identify regions of different variation and carry out integration over these subregions independently. In our Matlab programming, we relied in each subregion on *quadl*, a convenient adaptive implementation of the Gauss-Lobatto quadrature.

Figures 1(a) and 1(b) show influence curves of the MDPD functionals for $\alpha = 0.8, 1.1, 1.4$ and for $\gamma = 0.01$ and 0.05, respectively. They show that the gross error sensitivity is increasing when γ decreases towards 0. This remains consistent with our results on the efficiency of the estimators as the efficiency increases with γ decreasing towards 0. Hence, γ represents a parameter controlling a balance between the efficiency and robustness.

It may be interesting to note the multi-modal and non-monotonous character of the influence curves. The influence of the observations in a close vicinity of zero is close to zero. However next, in the middle zone, we observe two symmetric positive peaks of the influence function followed by a decay into negative values. The behavior of the influence curve seems very interesting in the external zones, where, by Corollary 1, it is returning towards zero via negative values. We suggest the interpretation of this feature of the influence curve as a shift of the value of the shape parameter towards zero caused by large observations. In the case of small values of α, huge observations abound in the alpha-stable sample, and so indeed, the presence of large observations may provide evidence towards a low value of α. The large observations should not be considered here as outliers



(a)



(b)

Figure 1: Influence functions of the MDPDE for the shape parameters α and selected values of $\gamma = 0.01, 0.05$, respectively.

and removed from the sample because they carry important information about the shape of the alpha-stable distribution.

5. Efficiency of the MDPDEs of the shape parameter

The problems of consistency and asymptotic normality of the MPDPEs in general cases have already been considered in Basu et al. (1998) and, under different set of assumptions, in Juárez and Schucany (2006). The conditions required in Juárez and Schucany (2006) are less restrictive and are met in the considered case of alpha-stable distributions.

By using Theorem 2 of Juárez and Schucany (2006) the convergence holds true if the minimizer in expression (4) is unique and the density function is continuous. Both conditions are met in our case and so, the conclusion on consistency follows.

By applying Theorem 5 of Juárez and Schucany (2006) or Theorem 2 of Basu et al. (1998) in the case of alpha-stable distributions and for the considered MDPDEs we infer the following results on the asymptotic distribution of the MPDPEs. We use these results to calculate theoretical asymptotic variances and asymptotic efficiencies of the MDPDEs presented in Table 1.

Proposition 2 If $Y_1, \cdots, Y_n$ are i.i.d random variables with a cumulative distribution function F and density f, then for $\hat{\alpha}_{\gamma,n} = \alpha_\gamma(\hat{F}_n)$, the empirical MDPDEs of the shape parameters $\alpha \in (0,2]$ of alpha-stable distributions, we have

$$\sqrt{n}\left(\hat{\alpha}_{\gamma,n} - \alpha_\gamma(F)\right) \longrightarrow \mathscr{N}\left(0, J_{\gamma,F}^{-1} M_{\gamma,F} J_{\gamma,F}^{-1}\right),$$

where $J_{\gamma,F}^{-1}$ is given by formula (7), ζ_F by formula (8) and

$$M_{\gamma,F}(\alpha_\gamma(F)) = \int u(y;\alpha)^2 f^{2\gamma}(y;\alpha) f(y) dy - \zeta_F^2.$$

Hence we have the following corollary suitable for our case.

Corollary 2 If F is a cumulative distribution function of a symmetric alpha-stable distribution F_α, then

$$\sqrt{n}\left(\hat{\alpha}_{\gamma,n} - \alpha_\gamma(F_\alpha)\right) \longrightarrow \mathscr{N}\left(0, AV(\alpha_\gamma(F_\alpha), F_\alpha)\right),$$

where $AV(\alpha_\gamma(F_\alpha), F_\alpha)$ is the asymptotic variance given by the sandwich formula

$$J_{\gamma,F_\alpha}^{-1} M_{\gamma,F_\alpha} J_{\gamma,F_\alpha}^{-1},$$

with

$$M_{\gamma,F_\alpha}(\alpha_\gamma(F_\alpha)) = \int f^{2\gamma-1}(y;\alpha)\left(\frac{\partial}{\partial\alpha} f(y;\alpha)\right)^2 dy - \zeta_{F_\alpha}^2,$$

$$\zeta_{F_\alpha} = \int f^\gamma(y;\alpha) \frac{\partial}{\partial\alpha} f(y;\alpha) dy,$$

and where J_{γ,F_α} is given by formula (10).

Equipped with the formulae of Corollary 2 for asymptotic variances $AV(\alpha_\gamma(F_\alpha), F_\alpha)$ one can now numerically calculate asymptotic efficiencies of the MDPDEs. We recall that the limiting case $\gamma = 0$ corresponds to the Kullback-Leibler discrepancy and that the Maximum Likelihood estimator coincides in this case with the MDPDEs, c.f. Basu et al. (1998). Hence, the asymptotic efficiency of the MDPDEs can be found by comparing the asymptotic variances $AV(F, \alpha_\gamma(F))$ and $AV(F, \alpha_0(F))$. This can be evaluated numerically, but great care needs to be taken in numerical evaluation of the related integrals. As we discussed in Section 2, an iterative procedure has to be implemented to guarantee high accuracy of the numerical values of the integrals.

In Table 1, we report the asymptotic relative efficiencies for the selected values of γ and for $\alpha \in \{0.1, (0.1), 2\}$. It can be seen that as the value of γ increases, the asymptotic relative efficiencies

Table 1: Asymptotic efficiencies of the MDPDEs for selected α's and γ's.

$\alpha \backslash \gamma$	0	0.01	0.02	0.05	0.1	0.5	1
0.1	1	0.9998	0.9998	0.9997	0.9993	0.9831	0.9170
0.2	1	0.9941	0.9758	0.8888	0.7574	0.4930	0.4168
0.3	1	0.9930	0.9736	0.8754	0.7112	0.4021	0.3375
0.4	1	0.9939	0.9775	0.8901	0.7237	0.3645	0.2849
0.5	1	0.9952	0.9821	0.9094	0.7553	0.3534	0.2589
0.6	1	0.9962	0.9859	0.9264	0.7892	0.3523	0.2470
0.7	1	0.9970	0.9888	0.9399	0.8200	0.3596	0.2443
0.8	1	0.9976	0.9909	0.9506	0.8459	0.3702	0.2432
0.9	1	0.9980	0.9925	0.9581	0.8672	0.3817	0.2474
1.0	1	0.9984	0.9937	0.9651	0.8865	0.3966	0.2511
1.1	1	0.9986	0.9947	0.9704	0.9016	0.4111	0.2533
1.2	1	0.9987	0.9959	0.9763	0.9196	0.4332	0.2588
1.3	1	0.9990	0.9957	0.9807	0.9323	0.4503	0.2587
1.4	1	0.9993	0.9965	0.9838	0.9414	0.4632	0.2536
1.5	1	0.9994	0.9973	0.9861	0.9486	0.4737	0.2438
1.6	1	0.9995	0.9978	0.9875	0.9550	0.4804	0.2279
1.7	1	0.9984	0.9980	0.9883	0.9588	0.4810	0.2038
1.8	1	0.9996	0.9971	0.9903	0.9636	0.4690	0.1674
1.9	1	0.9997	0.9987	0.9922	0.9658	0.4261	0.1117
2.0	1	0.9998	0.9992	0.9951	0.9830	$3.6E-11$	$2.17E-11$

decrease. This is not surprising, as we noted earlier, and shows that γ controls the trade-off between robustness and efficiency. In the case where the user is interested in retaining both asymptotic properties of these estimators, we recommend choosing positive values of γ that are less or equal to 0.05. We note that the asymptotic efficiencies for values of γ greater than 0.05 decrease significantly. These results show that the MDPDEs for $\gamma \leq 0.05$ are robust and simultaneously highly efficient as estimators of the shape parameter α.

6. Performance of the MDPDEs of α in Monte Carlo simulations

We conducted a variety of Monte Carlo type simulations to provide empirical evidence on the MDPDEs' efficiency, robustness, consistency and asymptotic normality. The sample sizes in our simulations were ranging from $n = 10$ to $n = 1000$.

We compared the performance of the MDPDEs with other existing estimators such as the Quantile, Empirical Characteristic Function (ECF), Fractional Lower Order Moment (FLOM) and Log-Moment estimators both in uncontaminated (Section 6.3) and contaminated (Section 6.4) models.

6.1. Numerical issues related to the MDPDEs

To simulate alpha-stable random variables we used the most popular and efficient Chambers-Mallows-Stuck (CMS) method, c.f. Chambers, Mallows and Stuck (1976) and Weron (1996). The MDPD methodology requires finding a root of equation (5) using an iterative algorithm starting with an initial guess for the root. Our extensive preliminary simulations showed that the final value of the estimator of α is not significantly affected by the choice of the initial values. Therefore we used the Quantile method to generate the starting values for all our simulations and data analysis presented in the paper.

It is interesting to note that small values of γ ($\gamma < 0.2$) required significantly less time to compute the estimators. We also observed that the computational time increases substantially as the value of γ gets closer to 1. Based on simulations carried out, we suggest using a fixed value of γ less than 0.05. This provides a good compromise between robustness and efficiency as noted in Table 1 and also remains relatively fast.

6.2. Performance of the MDPDEs of α for small sample sizes

To get a glimpse into a speed of convergence of properties the MDPDEs of α to the limiting values we performed for $\alpha = 0.1$ and $\alpha = 0.7$ simulations of $N = 500$ samples of sizes $n = 10, 20, 25, 50, 100, 200, 250$, respectively. For each sample we calculated the MDPDE of α. The averages, medians and the corresponding measures of deviations: mean absolute deviation (*Mean A Dev*), median absolute deviation (*Median A Dev*) and standard deviation (*Standard Dev*) are reported in Table 2. To facilitate comparison across sample sizes the reported in Table 2 measures of spread have been standardized by multiplying by the square root of the sample size $\sqrt{n}$. The obtained results show very good behavior of the MDPDE of α even for extremely small sample sizes like $n = 20$.

6.3. Comparison of the MDPDEs of α with other existing estimators in the case of uncontaminated model

We simulated 150 samples each of size $n = 1000$. For each sample, we obtained estimators of α using the MDPD method and, for the sake of comparison, by evaluating four other selected estimators: ECF, Quantile, FLOM and Log-Moment estimators. We fixed $\gamma = 0.01$ for the MDPD approach. The results of these simulation are summarized in Table 3.

Table 2: Performance of the MDPDEs of α for small sample sizes and $\gamma = 0.01$. The measures of spread have been standardized here by multiplying by the square root of the sample size $\sqrt{n}$.

$\alpha = 0.1$							
n	10	20	25	50	100	200	250
Average	0.1129	0.1051	0.1044	0.1023	0.1010	0.1007	0.1005
Median	0.1071	0.1018	0.1019	0.1010	0.1006	0.1000	0.1002
Mean A Dev	0.0753	0.0686	0.0677	0.0668	0.0620	0.0657	0.0645
Median A Dev	0.0584	0.0531	0.0559	0.0604	0.0537	0.0525	0.0540
Standard Dev	0.0967	0.0890	0.0865	0.0828	0.0789	0.0823	0.0799
$\alpha = 0.7$							
n	10	20	25	50	100	200	250
Average	0.8044	0.7480	0.7344	0.7170	0.7124	0.7028	0.7046
Median	0.7106	0.7123	0.7039	0.7047	0.7054	0.7024	0.7015
Mean A Dev	0.7834	0.6825	0.6516	0.5788	0.6433	0.5892	0.6015
Median A Dev	0.4723	0.5268	0.4679	0.4371	0.5594	0.5049	0.5269
Standard Dev	1.1076	0.8994	0.8874	0.7607	0.7984	0.7356	0.7491

Table 3: The mean estimators of α based 150 samples each of size $n = 1000$ using different estimation techniques. The parenthesis show the mean absolute deviations for these estimators.

α\Method	MDPDE	ECF	QUANTILE	FLOM	LOG MOMENTS
0.1	0.099895 (0.0025)	0.109613 (0.0084)	0.557741 (0.0219)	0.114831 (0.0054)	0.100031 (0.0038)
0.2	0.201195 (0.0043)	0.192562 (0.0109)	0.585209 (0.0091)	0.168696 (0.0116)	0.167026 (0.0076)
0.3	0.300572 (0.0073)	0.290332 (0.0125)	0.577200 (0.0049)	0.277101 (0.0160)	0.283993 (0.0120)
0.4	0.400349 (0.0097)	0.401523 (0.0145)	0.590919 (0.0031)	0.368809 (0.0203)	0.365289 (0.0162)
0.5	0.500778 (0.0120)	0.477735 (0.0195)	0.584516 (0.0023)	0.504140 (0.0264)	0.485576 (0.0209)
0.6	0.601665 (0.0158)	0.563039 (0.0226)	0.592018 (0.0207)	0.588951 (0.0318)	0.590621 (0.0258)
0.7	0.704305 (0.0194)	0.710061 (0.0252)	0.744606 (0.0271)	0.783938 (0.0396)	0.734887 (0.0323)
0.8	0.803472 (0.0227)	0.696219 (0.0266)	0.778418 (0.0277)	0.731437 (0.0477)	0.692154 (0.0376)
0.9	0.902889 (0.0242)	0.885927 (0.0284)	0.854224 (0.0356)	0.881350 (0.0571)	0.929367 (0.0476)
1.0	1.003325 (0.0280)	1.036830 (0.0341)	0.999243 (0.0332)	1.023626 (0.0642)	1.015373 (0.0548)
1.1	1.103277 (0.0308)	1.161730 (0.0328)	1.154472 (0.0390)	1.208372 (0.0840)	1.178021 (0.0697)
1.2	1.203009 (0.0332)	1.162077 (0.0368)	1.186217 (0.0414)	1.027965 (0.0931)	1.238859 (0.0808)
1.3	1.302350 (0.0357)	1.283592 (0.0359)	1.219519 (0.0416)	1.165643 (0.1229)	1.156782 (0.0942)
1.4	1.401837 (0.0375)	1.414603 (0.0396)	1.413116 (0.0455)	1.335447 (0.1366)	1.283951 (0.1126)
1.5	1.501344 (0.0383)	1.497591 (0.0413)	1.465904 (0.0514)	1.600245 (0.1712)	1.354119 (0.1326)
1.6	1.600250 (0.0377)	1.533427 (0.0423)	1.480808 (0.0559)	1.875616 (0.2003)	1.410342 (0.1553)
1.7	1.698957 (0.0360)	1.704310 (0.0392)	1.707026 (0.0643)	1.640730 (0.2561)	1.935429 (0.1670)
1.8	1.799019 (0.0342)	1.795019 (0.0386)	1.782433 (0.0768)	1.543741 (0.4942)	1.800046 (0.2158)
1.9	1.899767 (0.0280)	1.902605 (0.0309)	1.814127 (0.0709)	2.086989 (0.3334)	2.059261 (0.2477)

The observed relative absolute bias for α in the case of MDPDEs is definitely the lowest compared to the ECF, Quantile, FLOM and Log-Moment estimators. In particular, for $\alpha < 0.7$, the Quantile method performs extremely poorly compared to the other methods and we refer to Section 6 of Kangogo (2012) for more details. Weron (2001) also discussed the high bias of the loglog linear regression and Hill estimators of the index α and the implied confusing consequences for Market modeling in Finances.

6.4. Comparison of the robustness of the MDPDEs of α with other existing estimators in the contaminated case

The robustness of the MDPDEs of α can be investigated by observing how much these estimators change in the case of a slightly misspecified model. Clearly, a robust estimator should not be too much perturbed by the contaminating observations in the model. In our simulations, the distribution of these contaminating observations has been assumed to be known.

We investigated finite sample behavior of the MDPDEs of α at the contaminated model. We considered a mixture of two distributions both of which belong to the family of symmetric alpha-stable distributions. The benchmark distribution model is denoted by S_{α_0} and the contaminating distribution by S_{α_1}, where S_α stands for the probability distribution of a random variable Z given by formula (2). The mixture distribution model can be formulated as

$$(1-\varepsilon)S_{\alpha_0} + \varepsilon S_{\alpha_1}, \tag{12}$$

where ε was chosen from the set $\varepsilon \in \{0.01, 0.025, 0.05, 0.1\}$.

Let $M(S_{\alpha_0}, S_{\alpha_1}, \varepsilon)$ stand for the mixture model given by formula (12). Choosing α_0 and α_1 that are close to each other implies that also the two distributions lie in a vicinity of each other. We considered the following two mixture models:

$$M1: \quad M(S_{0.2}(1,0,0), S_{0.1}(1,0,0), \varepsilon) \quad \text{and} \quad M2: \quad M(S_{0.5}(1,0,0), S_{0.4}(1,0,0), \varepsilon).$$

We performed simulation studies for $M1$ and $M2$ to compare the robustness of the MDPDEs of α with the robustness of the other four estimators. We generated 500 contaminated samples each of size $n = 200$ with varying proportions of contamination $\varepsilon \in \{0.01, 0.025, 0.05, 0.1\}$. Based on these 500 samples, we computed using the five different estimation methods the mean absolute biases of the estimators of α for the two mixture models. The results are shown in Figures 2(a) and 2(b) respectively. The proportions of contamination are presented on the horizontal axis and the mean absolute biases of the considered estimators are shown on the vertical axes. The MDPD estimators, the ECF, the FLOM, the Quantiles and the Log-Moment are represented by a star, square, diamond, circle and plus signs, respectively.

In the considered mixture models, the mean absolute biases of the MDPDEs of α are the smallest compared to the other methods of estimation. In the case of Model $M1$, the Quantile estimator behaves very poorly. This is not surprising, as we noted earlier, that the Quantile method works reasonably well only for values of $\alpha \geq 0.7$. The ECF does better than FLOM, Quantiles and the Log Moments for Models $M1$ and $M2$.

To show how sensitive are the MDPDEs in the contaminated models to the tuning parameter γ and to the contaminating parameter ε we produce in Table 4 the mean estimators of the MDPDEs of

α for the reported two mixture models. The parentheses show the mean absolute deviation for these estimators. Similar conclusions can also be drawn for other mixture models which, for brevity, are not reported here.

Table 4: The mean estimators of the MDPDEs of α for mixture models $M1 = M(S_{0.2}(1,0,0), S_{0.1}(1,0,0), \varepsilon)$ and $M2 = M(S_{0.5}(1,0,0), S_{0.4}(1,0,0), \varepsilon)$. The parentheses show the mean absolute deviation for these estimators.

Model $\backslash\gamma$	0.01	0.02	0.05	0.1
$\varepsilon = 0.01$				
$M1$	0.1988 (0.0093)	0.1987 (0.0098)	0.2004 (0.0110)	0.1994 (0.0122)
$M2$	0.4999 (0.0270)	0.5000 (0.0283)	0.5027 (0.0311)	0.5002 (0.0318)
$\varepsilon = 0.025$				
$M1$	0.1955 (0.0096)	0.1957 (0.0100)	0.1958 (0.0113)	0.1947 (0.0118)
$M2$	0.4972 (0.0285)	0.4985 (0.0275)	0.4979 (0.0297)	0.4980 (0.0312)
$\varepsilon = 0.05$				
$M1$	0.1900 (0.0099)	0.1900 (0.0100)	0.1902 (0.0113)	0.1909 (0.0125)
$M2$	0.4952 (0.0284)	0.4936 (0.0270)	0.4967 (0.0290)	0.4964 (0.0306)
$\varepsilon = 0.1$				
$M1$	0.1797 (0.0095)	0.1809 (0.0109)	0.1805 (0.0116)	0.1778 (0.0139)
$M2$	0.4884 (0.0281)	0.4888 (0.0287)	0.4902 (0.0297)	0.4920 (0.0321)

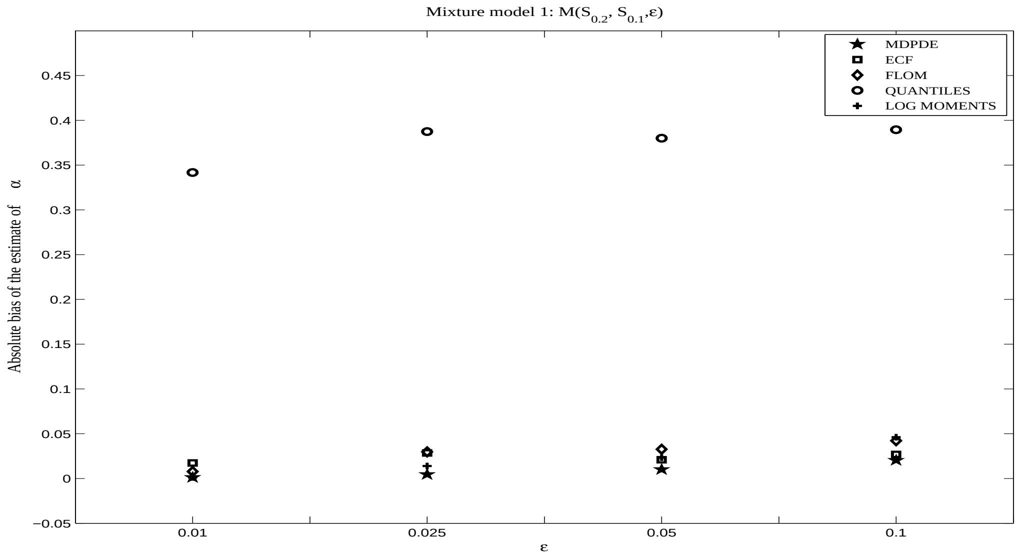
6.5. Dependence of the MDPDEs of α on the tuning parameter γ

Under the assumption that the considered model is correct, we conducted a simulation study to investigate how the MDPDEs of α depend on the tuning parameter γ. This study also gives some insight into the empirical measure of efficiency for the MDPDEs of α. As in the uncontaminated case before, we generated 500 samples each of size $n = 200$ from the S_α. For each value of γ in the set $\{0.01, 0.02, 0.05, 0.1, 0.5, 1.0\}$ we evaluated the MDPDEs $\hat{\alpha}_{\gamma,n}$ and reported in Table 5 their empirical means, empirical Standard Deviations (SD) and the square roots of the Mean Squared Errors (RMSEs).

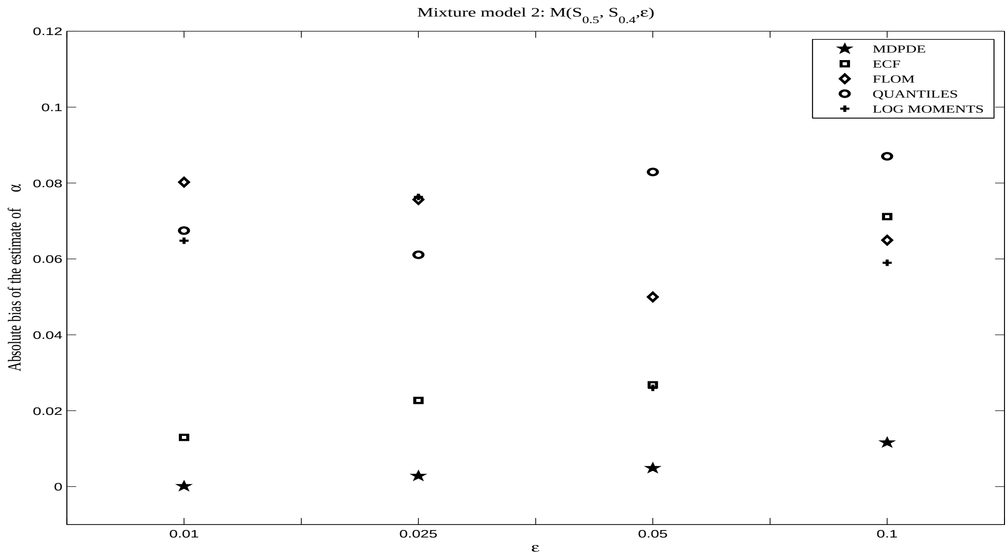
As expected, the small values of γ produced the smallest RMSEs and the estimated RMSEs increase as γ increase towards 1 for all chosen values of α. We conclude that fixed positive values of $\gamma \leq 0.05$ gave a reasonable compromise between robustness and efficiency. Let us note that these results are in agreement with the theoretical results on the efficiency of the MDPDEs, reported in Table 1. The results presented in Table 5 show that the size 500 of the considered samples is sufficient for the theoretical asymptotic properties to dominate and allow us to conclude about the excellent performance of these estimators.

6.6. Concluding remarks from the simulation results

Our simulation results confirm that when the assumed symmetric alpha-stable model is uncontaminated, then the MDPDEs of α are highly efficient and at the same time they retain the robustness



(a)



(b)

Figure 2: Mean absolute biases of the estimators of α obtained using different methods from mixture models with varying proportions of contaminations ε. For the MDPD method $\gamma = 0.01$.

Table 5: Theoretical asymptotic standard deviations (SD) and square roots of mean square errors (RMSE) of the MDPDEs of α computed for different values of γ.

	$\gamma = 0.01$	$\gamma = 0.02$	$\gamma = 0.05$	$\gamma = 0.1$	$\gamma = 0.5$	$\gamma = 1.0$
$\alpha = 0.2$						
$\bar{\hat{\alpha}}_{500}$	0.2006	0.2016	0.2013	0.2016	0.2032	0.2027
SD	0.0114	0.0118	0.0130	0.0156	0.0183	0.0199
RMSE	0.0114	0.0120	0.0131	0.0157	0.0185	0.0200
$\alpha = 0.5$						
$\bar{\hat{\alpha}}_{500}$	0.5008	0.5029	0.5000	0.5033	0.5011	0.5033
SD	0.0329	0.0352	0.0371	0.0423	0.0371	0.0423
RMSE	0.0329	0.0353	0.0371	0.0425	0.0371	0.0425
$\alpha = 0.8$						
$\bar{\hat{\alpha}}_{500}$	0.8040	0.8025	0.8013	0.8059	0.7942	0.8102
SD	0.0604	0.0583	0.0625	0.0670	0.1010	0.1217
RMSE	0.0605	0.0584	0.0625	0.0672	0.1011	0.1222
$\alpha = 1.2$						
$\bar{\hat{\alpha}}_{500}$	1.2102	1.2147	1.2155	1.2015	1.1986	1.2129
SD	0.0934	0.0952	0.0910	0.0968	0.1408	0.1893
RMSE	0.0940	0.0963	0.0923	0.0968	0.1408	0.1898
$\alpha = 1.5$						
$\bar{\hat{\alpha}}_{500}$	1.5041	1.5097	1.5054	1.5082	1.4937	1.4951
SD	0.1029	0.1041	0.0980	0.1059	0.1485	0.2098
RMSE	0.1030	0.1045	0.0981	0.1062	0.1487	0.2098
$\alpha = 1.8$						
$\bar{\hat{\alpha}}_{500}$	1.7980	1.8041	1.8076	1.8011	1.7923	1.7531
SD	0.0896	0.0889	0.0925	0.0981	0.1285	0.2106
RMSE	0.0896	0.0890	0.0928	0.0981	0.1287	0.2157

properties. Our simulations based on mixture models also confirm that these estimators are robust. The proportions of the contaminations we used were from the set $\varepsilon = \{0.01, 0.025, 0.05, 0.1\}$. For the obtained MDPDEs of α, the reported mean absolute biases were relatively small.

The MDPDEs perform particularly well in comparison with the four existing estimators when the chosen value of γ is less than 0.05. We found serious problems related to numerical convergence for some of the methods we considered like the FLOM and the Quantile methods. A good example of this type of problems is the method based on quantiles where for $\alpha < 0.7$, the method breaks down. It remains an open question what the breakdown points are for the considered five estimation methods in the case of stable distributions.

Time needed for evaluation of the MDPDEs of α depends significantly on the parameter α of the stable distribution from which the sample was generated and varies from several seconds to a few minutes. The bottle neck is in the precise numerical evaluation of the alpha-stable probability density function and its derivatives appearing in equation (5). The region in the vicinity of the value of $\alpha = 1$ and also at some regions in the vicinity of $x = 0$ are particularly difficult and time consuming. We refer to Section 2 of Kangogo (2012) for more details and discussion of the implemented numerical methods.

A complete set of Matlab programs developed and used in this project is available upon request from the authors.

7. Conclusions

We have introduced and explored MDPDE, a new efficient and robust estimator of the shape parameter of alpha-stable distributions. We found that these MDPDEs exhibit high efficiency and good robustness properties for small γ.

Our simulation studies reported in Section 6 provide a strong support to our theoretical findings. In the case of samples contaminated with probability distributions having heavier tails, the MDPDEs showed significantly lower mean absolute biases compared to ECF, Quantile, FLOM and Log Moments.

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