FISHER'S INFORMATION AND THE CLASS OF f-DIVERGENCES BASED ON EXTENDED ARIMOTO'S ENTROPIES

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Abstract: Csiszár (1963) and Ali and Silvey (1966) introduced the concept of f-divergence, which is a measure of deviation of two probability distributions, like e.g. Pearson's χ^2-deviation, which is given in terms of a convex function f defined on $[0,\infty)$. Vajda (1972, 1973) extended Pearson's χ^2-deviation to the family of the so-called χ^α-divergences, $\alpha \in [1,\infty)$.

Österreicher and Vajda (2003) introduced a family of f-divergences closely linked to the class

$$h_\alpha(t) = \begin{cases} \frac{1}{1-\alpha}[1-(t^{1/\alpha}+(1-t)^{1/\alpha})^\alpha] & \text{if} \quad \alpha \in (0,\infty)\backslash\{1\} \\ -[t\ln t + (1-t)\ln(1-t)] & \text{if} \quad \alpha = 1 \\ \min(t, 1-t) & \text{if} \quad \alpha = 0 \end{cases}$$

of Arimoto's entropies (1971). Vajda (2009) extended the corresponding family I_{φ_α} of f-divergences to all $\alpha \in \mathbb{R}$.

In Section 2 we consequently extend Arimoto's class of entropies to all $\alpha \in \mathbb{R}$. Theorem 3 in Section 4 relates the family I_{φ_α} of f-divergences to Fisher's Information in a limiting way for all $\alpha \in \mathbb{R}$ except for those from a certain neighbourhood of $\alpha = 0$. Its proof relies on an inequality of the form

$$\left| I_{\varphi_\alpha}(Q,P) - \frac{\varphi_\alpha''(1)}{2} \cdot \chi^2(Q,P) \right| \leq c_\alpha \cdot \chi^3(Q,P) \,,$$

which may be interesting also in its own right. This family of inequalities and its basic analytic counterpart are stated in Section 3. The corresponding proof is postponed to the Appendix.

Section 2 was stimulated by Vajda's paper of 2009 and Section 4 considerably by his paper of 1973.

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1. Introduction

Let $(\Omega, \mathscr{A})$ be a nondegenerate measurable space (i.e. $|\mathscr{A}| > 2$ and hence $|\Omega| > 1$), μ a σ-finite measure on $(\Omega, \mathscr{A})$ and let $\mathscr{P}(\Omega, \mathscr{A})$ be the set of probability distributions on $(\Omega, \mathscr{A})$ dominated by μ. Furthermore, let $\mathscr{F}$ be the set of convex functions $f : [0,\infty) \mapsto (-\infty, \infty]$ which are finite on $(0,\infty)$ and continuous on $[0,\infty)$. In addition, let the function $f^* \in \mathscr{F}$ be defined by

$$f^*(u) = u \cdot f(1/u) \quad \text{for } u \in (0,\infty). \tag{1}$$

Remark 1 By setting $0 \cdot f(v/0) = \begin{cases} 0 & \text{for} \quad v = 0 \\ v \cdot f^*(0) & \text{for} \quad v > 0 \end{cases}$ for all $f \in \mathscr{F}$, it holds

$$x \cdot f^*(y/x) = y \cdot f(x/y) \quad \text{for all } x, y \in [0,\infty).$$

Definition 1 The function $f^* \in \mathscr{F}$ defined by (1) is called the $*$-conjugate function of f. A function $f \in \mathscr{F}$ which satisfies $f^* \equiv f$ is called $*$-self conjugate.

Definition 2 (cf. Csiszár, 1963; Ali and Silvey, 1966) Let $P, Q \in \mathscr{P}(\Omega, \mathscr{A})$. Then

$$I_f(Q,P) = \int f(\frac{q}{p}) \cdot p d\mu$$

is called the f-divergence of Q and P. (As usual, $p = \frac{dP}{d\mu}$ and $q = \frac{dQ}{d\mu}$ are the Radon-Nikodym-derivatives of P and Q with respect to μ.)

Remark 2 (a) It holds $I_f(Q,P) \geq f(1)\ \forall P, Q \in \mathscr{P}(\Omega, \mathscr{A})$.
(b) Provided that the function $f \in \mathscr{F}$ is $*$-self conjugate, the corresponding f-divergence is symmetric, i.e. it satisfies $I_f(Q,P) = I_f(P,Q)\ \ \forall P, Q \in \mathscr{P}(\Omega, \mathscr{A})$.

In the sequel we restrict ourselves - without loss of generality - to the subset $\mathscr{F}_1$ of convex functions $f \in \mathscr{F}$ which satisfy $f(1) = 0$, are strict convex for $u = 1$ and satisfy $f(u) \geq 0\ \forall\, u \in [0,\infty)$. The subset of functions $f \in \mathscr{F}_1$, which are $*$-self conjugate, is denoted by $\tilde{\mathscr{F}}_1$.
First let us consider the so-called **Class of Power Divergences**

$$I_\alpha(Q,P) = \begin{cases} \frac{1}{\alpha(1-\alpha)}(1 - \int p^{1-\alpha} q^\alpha d\mu) & \text{for} \quad \alpha \in \mathbb{R}\backslash\{0,1\} \\ \int q \ln(\frac{q}{p}) d\mu & \text{for} \quad \alpha = 1 \\ \int p \ln(\frac{p}{q}) d\mu & \text{for} \quad \alpha = 0\ , \end{cases}$$

which is given in terms of the following class of functions $\psi_\alpha \in \mathscr{F}_1$

$$\psi_\alpha(u) = \begin{cases} \frac{\alpha \cdot u + 1 - \alpha - u^\alpha}{\alpha(1-\alpha)} & \text{for} \quad \alpha \in \mathbb{R}\backslash\{0,1\} \\ 1 - u + u \ln u & \text{for} \quad \alpha = 1 \\ u - 1 - \ln u & \text{for} \quad \alpha = 0\ , \end{cases}$$

for which ψ_1 and ψ_0 are limiting cases.
For the corresponding $*$-conjugate functions ψ_α^* it holds $\psi_\alpha^* = \psi_{1-\alpha}\ \forall \alpha \in \mathbb{R}$. Hence the only parameter, for which I_α is symmetric, is $\alpha = \frac{1}{2}$.

Special cases. The class of f-divergences defined in terms of the functions $\psi_\alpha \in \mathscr{F}_1$ include the following well-known divergences.

$\alpha = 1$: Kullback-Leibler-Divergence $I(Q\|P)$

$\alpha = 0$: Kullback-Leibler-Divergence $I(P\|Q)$

$\alpha = 2$: Pearson's χ^2-Divergence ($2 \cdot \psi_2(u) = (u-1)^2$)

$\alpha = 1/2$: Squared Hellinger Distance ($\psi_{1/2}(u)/2 = (\sqrt{u}-1)^2$)

$\alpha = -1$: Neyman-Divergence ($2 \cdot \psi_{-1}(u) = (u-1)^2/u$) .

The class of power divergences has a long history. It was introduced for $\alpha > 0$,
$\alpha \neq 1$, by Rényi (1961) and extended to $\mathbb{R}$ by Liese and Vajda (1987). The Havrda-Charvát class and the Cressie-Read class are equal to this class, however with different parametrizations. The former, which goes back to Havrda and Charvát (1967), is defined in terms of $\psi_{(\beta+1)/2}(u)$, $\beta \in \mathbb{R}$. The latter, introduced by Cressie and Read (1984), is defined in terms of $\psi_{\lambda+1}(u)$, $\lambda \in \mathbb{R}$.
The symmetric version of an f-divergence is given in terms of $\tilde{f} = (f + f^*)/2$, $\tilde{f} \in \tilde{\mathscr{F}}_1$. Therefore, the symmetric version of the class of power divergences is given in terms of the functions

$$\tilde{\psi}_\alpha(u) = \frac{1}{2}\begin{cases} \frac{u+1-(u^\alpha+u^{1-\alpha})}{\alpha(1-\alpha)} & \text{for} \quad \alpha \in \mathbb{R}\backslash\{0,1\} \\ (u-1)\ln u & \text{for} \quad \alpha \in \{0,1\}\ . \end{cases}$$

As shown by Csiszár and Fischer (1962) the parameters $\alpha \in (0,1)$ provide the distances $[I_{\tilde{\psi}_\alpha}(Q,P)]^{\min(\alpha,1-\alpha)}$.

Remark 3 In order that an f-divergence $I_f(Q,P)$ allows for a metric it is required that

$$I_f(Q,P) \geq 0 \ \ \forall\, P,Q \in \mathscr{P}(\Omega,\mathscr{A}),$$

with equality iff $Q = P$, and that symmetry

$$I_f(Q,P) = I_f(P,Q) \ \ \forall\, P,Q \in \mathscr{P}(\Omega,\mathscr{A})$$

holds. These two conditions are guaranteed by the assumption $f \in \mathscr{F}_1$ and by that of its $*$-self conjugacy and, consequently, by $f \in \tilde{\mathscr{F}}_1$. The latter and the further requirement

$$f(0) < \infty$$

are necessary conditions in order to guarantee that a certain power $[I_f(Q,P)]^\alpha$, $\alpha \in (0,1]$, is a distance. For a detailed discussion of this topic we refer to Section 3.4 of the paper by Österreicher (2013). By the way, for $f \in \mathscr{F}_1$ the condition $f^*(0) = \infty$ is a sufficient condition in order to guarantee at least one f-projection. For a detailed discussion of f-projections we refer e.g. to the outstanding master's thesis by Kafka (1995).

Remark 4 For the class of power divergences it holds

$$\tilde{\psi}_\alpha(0) = \begin{cases} \frac{1}{2\alpha(1-\alpha)} & \text{for} \quad \alpha \in (0,1) \\ \infty & \text{for} \quad \alpha \in (-\infty,0] \cup [1,\infty) \end{cases} \quad \text{and} \quad \psi^*_\alpha(0) = \begin{cases} \infty & \text{for} \quad \alpha \in [1,\infty) \\ \frac{1}{1-\alpha} & \text{for} \quad \alpha \in (-\infty,1). \end{cases}$$

In this paper we will be interested in specfic results on the f-divergences given in terms of the following class of functions $\varphi_\alpha \in \tilde{\mathscr{F}}_1$

$$\varphi_\alpha(u) = \begin{cases} \frac{sgn(\alpha)}{1-\alpha}((1+u^{1/\alpha})^\alpha - 2^{\alpha-1}\cdot(1+u)) & \text{for} \quad \alpha \in \mathbb{R}\backslash\{0,1\} \\ (1+u)\ln 2 + u\ln u - (1+u)\ln(1+u) & \text{for} \quad \alpha = 1 \\ |u-1|/2 & \text{for} \quad \alpha = 0\ , \end{cases} \tag{2}$$

for which φ_1 and φ_0 are limiting cases. These were originally introduced and studied for nonnegative parameters α by Österreicher and Vajda (2003) and extended to $\alpha \in \mathbb{R}$ by Vajda (2009). For a detailed discussion see Section 5 of the paper by Österreicher (2013). According to the latter we call the corresponding class of divergences the **Class of Perimeter-Type Divergences**.

Special cases. The class of f-divergences defined in terms of the functions $\varphi_\alpha \in \tilde{\mathscr{F}}_1$ include the following well-known divergences.

$\alpha = 0$: Total Variation Distance ($\varphi_0(u) = |u-1|/2$)
$\alpha = 2$: Squared Hellinger Distance ($\varphi_2(u) = (\sqrt{u}-1)^2$)
$\alpha = 1$: Symmetrized Kullback-Leibler-Divergence
$\alpha = 1/2$: Squared Perimeter Distance ($\varphi_{1/2}(u)/2 = \sqrt{1+u^2} - (1+u)/\sqrt{2}$)
$\alpha = -1$: Squared Puri-Vincze Distance ($4\cdot\varphi_{-1}(u) = \frac{1}{2}\frac{(u-1)^2}{1+u}$) .

Remark 5 For the class of divergences given by $\varphi_\alpha \in \tilde{\mathscr{F}}_1,\ \alpha \in \mathbb{R}$, it holds

$$\varphi_\alpha(0) = \begin{cases} \frac{1-2^{\alpha-1}}{1-\alpha} & \text{for} \quad \alpha \in (0,\infty)\backslash\{1\} \\ \ln 2 & \text{for} \quad \alpha = 1 \\ \frac{1}{(1-\alpha)2^{1-\alpha}} & \text{for} \quad \alpha \in (-\infty, 0]\ . \end{cases}$$

Vajda (2009) proved that, besides of $\|Q-P\|/2$ for $\alpha = 0$,

$$[I_{\varphi_\alpha}(Q,P)]^{1/\max(\alpha,2)} \text{ for } \alpha \in \mathbb{R}\backslash\{0\}$$

are distances.

2. Arimoto's Entropies and their Extension

In the following we state a connection between an entropy h of a discrete probability distribution with two states and a convex function $f \in \tilde{\mathscr{F}}_1$ which is basic for the definition of the corresponding f-divergence.

Definition 3 Let $P = (t, 1-t),\ t \in [0,1]$ be a discrete probability distribution with two states. Then a concave function

$$h : [0,1] \mapsto \mathbb{R}\ ,$$

which is symmertic with respect to $t = \frac{1}{2}$ and which satisfies $h(0) = h(1) = 0$ and $h(\frac{1}{2}) \in (0,\infty)$ is - according to $h(t) =: H(P) = H((t,1-t)) = H((1-t,t))$ - called entropy of P or, simply, entropy.

The proof of the following statement, which relies on the observation of $\varphi(0) = h(\frac{1}{2})$, is straightforward.

Proposition 1 Provided that h is the entropy of a distribution with two states, then the function $\varphi : [0,\infty) \mapsto \mathbb{R}$ defined in terms of

$$\varphi(u) = (1+u)(h(1/2) - h(\frac{u}{1+u})), \;\; u \in [0,\infty)$$

is an element of $\tilde{\mathscr{F}}_1$. On the other hand, if $\varphi \in \tilde{\mathscr{F}}_1$ obeys these properties, then the function h defined by

$$h(t) = \varphi(0) - (1-t)\varphi(\frac{t}{1-t}), \;\; t \in [0,1] \;, \tag{3}$$

is an entropy.
As shown by Österreicher and Vajda (2003), Arimoto's (1971) class of entropies, which is given by

$$h_\alpha(t) = \begin{cases} \frac{1}{1-\alpha}[1-(t^{1/\alpha}+(1-t)^{1/\alpha})^\alpha] & \text{if} \quad \alpha \in (0,\infty)\backslash\{1\} \\ -[t\ln t + (1-t)\ln(1-t)] & \text{if} \quad \alpha = 1 \\ \min(t, 1-t) & \text{if} \quad \alpha = 0 \;, \end{cases} \tag{4}$$

generates our class of f-divergences for the parameters $\alpha \in [0,\infty)$. As Vajda (2009) extented the class of φ_α to all $\alpha \in \mathbb{R}$, Arimoto's class of entropies can be also extended to all $\alpha \in \mathbb{R}$. The corresponding functions h for negative $\alpha = -k$, $k \in (0,\infty)$, is owing to (3) given by

$$h_{-k}(t) = \frac{1}{1+k}\frac{t(1-t)}{(t^{1/k}+(1-t)^{1/k})^k} \;, \;\; t \in [0,1] \;.$$

Example For $\alpha = -1$ and $\alpha = -2$ the corresponding functions h are

$$h_{-1}(t) = \frac{1}{2}t(1-t) \quad \text{and} \quad h_{-2}(t) = \frac{t(1-t)}{3(1+2\sqrt{t(1-t)})},$$

respectively.

Remark 6 (Arimoto's representation) Let $\mathscr{P}_2 = \{Q = (s, 1-s) : s \in [0,1]\}$ and $P = (t, 1-t) \in \mathscr{P}_2$. Then Arimoto's entropies are defined in terms of the functions of uncertainty

$$\psi_\alpha(s) = \begin{cases} \frac{1-s^{1-\alpha}}{1-\alpha} & \text{for} \quad \alpha \in [0,\infty)\backslash\{1\} \\ -\ln s & \text{for} \quad \alpha = 1 \end{cases}$$

as follows: Let

$$\begin{aligned} H_{\psi_\alpha}(Q,P) \quad &: \quad = t \cdot \psi_\alpha(s) + (1-t)\cdot\psi_\alpha(1-s) \\ &= \quad \begin{cases} \frac{1-(t\cdot s^{1-\alpha}+(1-t)\cdot(1-s)^{1-\alpha}}{1-\alpha} & \text{for} \quad \alpha \in [0,\infty)\backslash\{1\} \\ -(t\ln s + (1-t)\ln(1-s)) & \text{for} \quad \alpha = 1 \;. \end{cases} \end{aligned}$$

Then the entropy given in (4) equals

$$h_\alpha(t) = H_{\psi_\alpha}(Q^*_\alpha, P) = \min\{H_{\psi_\alpha}(Q,P) : Q \in \mathscr{P}_2\}$$

with $Q^*_\alpha = (s_\alpha(t), 1-s_\alpha(t))$ and

$$s_\alpha(t) = \left\{ \frac{t^{1/\alpha}}{t^{1/\alpha}+(1-t)^{1/\alpha}} \quad \text{for} \quad \alpha \in (0,\infty) \right.$$

and

$$s_0(t) = \begin{cases} 1_{(1/2,1]}(t) & \text{for} \quad t \in [0,1]\backslash\{1/2\} \\ s \in [0,1] & \text{for} \quad t = 1/2 \ . \end{cases}$$

In all statements the cases $\alpha = 1$ and $\alpha = 0$ are limiting cases.

The appropriate representation of our new class of entropies for $\alpha \in (-\infty, 0)$ or $\alpha = -k,\ k \in (0,\infty)$ is given as follows.

Theorem 1 Let the corresponding functions of uncertainty be

$$\psi_{-k}(s) = \frac{(1-s)^{1+k}}{1+k},\ s \in [0,1]$$

and

$$\begin{aligned} \bar{h}_{-k}(t,s) &= H_{\psi_{-k}}(Q,P) = t \cdot \psi_{-k}(s) + (1-t)\cdot \psi_{-k}(1-s) \\ &= \frac{t\cdot(1-s)^{1+k} + (1-t)\cdot s^{1+k}}{1+k},\ t,s \in [0,1]. \end{aligned}$$

Then

$$\begin{aligned} h_{-k}(t) &= H_{\psi_{-k}}(Q^*_{-k}, P) = \min\{H_{\psi_{-k}}(Q,P) : Q \in \mathscr{P}_2\} \\ &= \frac{1}{1+k}\frac{t(1-t)}{(t^{1/k}+(1-t)^{1/k})^k} \end{aligned}$$

with $Q^*_{-k} = (s_{-k}(t), 1-s_{-k}(t))$ and $s_{-k}(t) = \frac{t^{1/k}}{t^{1/k}+(1-t)^{1/k}}$, $t \in [0,1]$.

Proof. It is a well-known fact that the envelope of the family of the linear functions

$$t \mapsto \bar{h}_{-k}(t,s),\ \ s \in [0,1]$$

is generated by setting

$$\frac{\partial}{\partial s}\bar{h}_{-k}(t,s) = 0\ .$$

This yields

$$t = \frac{s^k}{s^k+(1-s)^k} \quad \text{or, equivalently,} \quad s_{-k}(t) = \frac{t^{1/k}}{t^{1/k}+(1-t)^{1/k}}\ .$$

Then, owing to

$$(1-t)\cdot s^k_{-k}(t) = \frac{t(1-t)}{(t^{1/k}+(1-t)^{1/k})^k} = t\cdot(1-s_{-k}(t))^k$$

and $s_{-k}(t) + 1 - s_{-k}(t) = 1$ it holds

$$\begin{aligned} (1+k)\cdot h_{-k}(t) &= (1+k)\cdot \bar{h}_{-k}(t, s_{-k}(t)) = (1-t)\cdot s^{1+k}_{-k}(t) + t\cdot(1-s_{-k}(t))^{1+k} \\ &= (1-t)\cdot s^k_{-k}(t)\cdot s_{-k}(t) + t\cdot(1-s_{-k}(t))^k\cdot(1-s_{-k}(t)) \\ &= \frac{t\cdot(1-t)}{(t^{1/k}+(1-t)^{1/k})^k}\cdot(s_{-k}(t)+1-s_{-k}(t)) = \frac{t\cdot(1-t)}{(t^{1/k}+(1-t)^{1/k})^k}\ . \end{aligned}$$

■

3. Upper Bounds for the Second Order Taylor Approximation

Let $\chi^\alpha(Q,P)$ be the f-divergence given by the convex function $f_\alpha(u) = |u-1|^\alpha$, $\alpha \in [1,\infty)$. [2] Then $\chi^1(Q,P) \equiv 2 \cdot I_{\varphi_0}(Q,P)$, i.e. the former is twice the Total Variation Distance, and

$$\chi^2(Q,P) = \int \frac{(q-p)^2}{p} d\mu$$

is Pearson's χ^2-Divergence. Furthermore, the f-divergence of the convex function $\varphi(u) = u\ln u + 1 - u$ is the so-called Kullback-Leibler- or I-Divergence

$$I(Q\|P) = \int \ln(\frac{q}{p}) \cdot q d\mu \ .$$

Motivation 1 Owing to $\varphi(u) \simeq \frac{1}{2} \cdot (u-1)^2$, it is a well-kown fact that the second order Taylor approximation of the I-divergence is $\frac{1}{2} \cdot \chi^2(Q,P)$ (cf. Jeffreys, 1946). In addition, the inequality

$$\left| I(Q\|P) - \frac{1}{2} \cdot \chi^2(Q,P) \right| \leq \frac{1}{2} \cdot \chi^3(Q,P) \ ,$$

which is useful when comparing Maximum-Likelihood-Estimation and Minimum χ^2-Estimation, holds true. The latter can be shown easily by checking $\left|\varphi(u) - \frac{1}{2} \cdot (u-1)^2\right| \leq \frac{1}{2} \cdot |u-1|^3 \quad \forall\, u \in [0,\infty)$.

In order to motivate Lemma 1 let us consider two special cases which allow for very easy ad hoc proofs of those inequalities on which the inequalities of the above type are based.

Proposition 2 (Special Cases $\alpha = -1$ and $\alpha = 2$) Let

$$\Delta_\alpha(u) = \varphi_\alpha(u) - \frac{\varphi''_\alpha(1)}{2} \cdot (u-1)^2, \ \ \alpha \in \{-1, 2\}.$$

Then

$$|\Delta_\alpha|(u) \leq c_\alpha \cdot |u-1|^3 \quad \text{with} \quad c_{-1} = \frac{1}{16} \quad \text{and} \quad c_2 = \frac{3}{4} \ .$$

Proof. Case $\alpha = -1$: Owing to $\frac{\varphi''_{-1}(1)}{2} = \frac{1}{16}$ it holds

$$\Delta_{-1}(u) = \frac{1}{8}\frac{(u-1)^2}{1+u} - \frac{1}{16}(u-1)^2 = \frac{1}{16} \cdot \frac{(1-u)^3}{1+u}$$

and hence

$$|\Delta_{-1}|(u) \leq \frac{1}{16} \cdot |u-1|^3 \ .$$

Case $\alpha = 2$: Owing to $\frac{\varphi''_2(1)}{2} = \frac{1}{4}$ and $\sqrt{u} - 1 = \frac{u-1}{\sqrt{u}+1}$ it holds

$$\begin{aligned} \Delta_2(u) &= (\sqrt{u}-1)^2 - \frac{1}{4}(u-1)^2 = \frac{1}{4} \cdot (4\frac{(u-1)^2}{(1+\sqrt{u})^2} - (u-1)^2) \\ &= \frac{1}{4} \cdot \frac{(u-1)^2(1-\sqrt{u})(3+\sqrt{u})}{(1+\sqrt{u})^2} = \frac{1}{4} \cdot (1-u)^3 \cdot \frac{3+\sqrt{u}}{(1+\sqrt{u})^3} \end{aligned}$$

[2] This class of f-divergences was introduced and investigated by Vajda (1972, 1973).

and - since the last factor is monotone decreasing - consequently

$$|\Delta_{-1}|(u) \leq \frac{3}{4} \cdot |u-1|^3 \ .$$

■

Now we state Lemma 1 which immediately yields the upper bounds for the second order Taylor approximations of the family $I_{\varphi_\alpha}(Q,P),\ \alpha \in \mathbb{R}$ of f-divergences considered in this paper for all α except for those from a certain neighbourhood $(-k_{\min}, \alpha_{\min})$ of $\alpha = 0$.

Lemma 1 Let $\varphi_\alpha,\ \alpha \in \mathbb{R}$ be the family of convex functions given in Section 1 and

$$\Delta_\alpha(u) = \varphi_\alpha(u) - \frac{\varphi''_\alpha(1)}{2} \cdot (u-1)^2 \quad \text{for } \alpha \neq 0\ .$$

Furthermore, let the limits $k_{\min}$ ($\simeq 0.14539$) and $\alpha_{\min}$ ($\simeq 0.1337$) be as defined in Proposition 6_{-k} and Proposition 6_α. Then for all $\alpha \in (-\infty, -k_{\min}] \cup [\alpha_{\min}, \infty)$ it holds

$$|\Delta_\alpha|(u) \leq c_\alpha \cdot |u-1|^3 \quad \forall\, u \in [0,\infty)\ .$$

The positive constants

$$c_\alpha = \max(c_\alpha^{(1)}, c_\alpha^{(0)})$$

are finite and, for a specific α, the maximum of the best possible upper bounds $c_\alpha^{(1)}$, $c_\alpha^{(0)}$ achievable on the interval $[1,\infty)$ and $[0,1)$ respectively.

Theorem 2 Let $\mathscr{P}(\Omega,\mathscr{A})$ be as given in Section 1. Then the family $I_{\varphi_\alpha},\ \alpha \in (-\infty,-k_{\min}] \cup [\alpha_{\min},\infty)$ obeys the inequalities

$$\left| I_{\varphi_\alpha}(Q,P) - \frac{\varphi''_\alpha(1)}{2} \cdot \chi^2(Q,P) \right| \leq c_\alpha \cdot \chi^3(Q,P) \quad \forall\, P,Q \in \mathscr{P}(\Omega,\mathscr{A})\ .$$

The functions φ_α behave substantially differently for values u close to $u=0$ when the sign of α is changed. Therefore, in order to state the bounds $c_\alpha^{(1)}$ and $c_\alpha^{(0)}$ it is appropriate to distinguish the cases

$\alpha \in (0,\infty)$ and

$\alpha \in (-\infty,0)$ with $\alpha = -k,\ k \in (0,\infty)$.

Cases $\alpha \in (0,\infty)$:

Interval $[1,\infty)$: ($\varphi_\alpha(0) = \frac{1-2^{\alpha-1}}{1-\alpha},\ \frac{\varphi''_\alpha(1)}{2!} = \frac{2^{\alpha-3}}{\alpha},\ \frac{|\varphi'''_\alpha(1)|}{3!} = \frac{2^{\alpha-4}}{\alpha}$)

$$\begin{aligned} \Delta_\alpha(u) &= \varphi_\alpha(u) - \frac{\varphi''_\alpha(1)}{2!}(u-1)^2 \\ &= \frac{1}{1-\alpha}((1+u^{1/\alpha})^\alpha - 2^{\alpha-1}(1+u)) - \frac{2^{\alpha-3}}{\alpha}(u-1)^2\ ,\ \alpha \in (0,\infty)\backslash\{1\}, \end{aligned}$$

respectively

$$\begin{aligned} \Delta_1(u) &= \varphi_1(u) - \frac{\varphi''_1(1)}{2}(u-1)^2 \\ &= (1+u)\ln 2 + u\ln u - (1+u)\ln(1+u) - \frac{1}{4}(u-1)^2 \end{aligned}$$

and hence

$$\Delta_\alpha(0) = \begin{cases} \frac{8\alpha-2^\alpha(1+3\alpha)}{8\alpha(1-\alpha)} & \text{for} \quad \alpha \in (0,\infty)\backslash\{1\} \\ \ln 2 - \frac{1}{4} & \text{for} \quad \alpha = 1 \ . \end{cases}$$

Let

$$c_a^{(1)} = \begin{cases} \frac{|\varphi_\alpha'''(u_\alpha)|}{3!} & \text{for} \quad \alpha \in (0, \frac{1}{3}) \\ \frac{|\varphi_\alpha'''(1)|}{3!} & \text{for} \quad \alpha \in [\frac{1}{3}, \infty) \end{cases}$$

with

$$\varphi_\alpha'''(u) = \frac{1}{\alpha^2}(1+u^{1/\alpha})^{\alpha-3} u^{1/\alpha-3} \cdot (1-2\alpha-(1+\alpha)u^{1/\alpha}) \ ,$$

$$u_\alpha = \left(\frac{1}{2(1+2\alpha)}(4-7\alpha+\sqrt{\frac{(2-\alpha)(6-11\alpha-\alpha^2)}{1+\alpha}})\right)^\alpha \ , \ \alpha \in (0,\alpha^*] \qquad (5_\alpha)$$

$\alpha^* = \frac{1}{2}(\sqrt{145}-11) \simeq 0.52080$ and $\lim_{\alpha\searrow 0} u_\alpha = 1, \ u_{\alpha^*} = (\frac{\alpha^*}{6})^{\alpha^*}$. Then for all $\alpha \in (0,\infty)$ it holds

$$|\Delta_\alpha|(u) \le c_a^{(1)} \cdot |u-1|^3 \quad \forall \, u \in [1,\infty) \ .$$

Remark 7 Condition $\varphi_\alpha''''(1) = 0$ is satisfied for $\alpha = \frac{1}{3}$.
Interval $[0,1)$: Let

$$c_\alpha^{(0)} = \begin{cases} \frac{|\varphi_\alpha'''(1)|}{3!} & \text{for} \quad \alpha \in [\alpha_{\min}, \frac{1}{3}] \\ \frac{|\varphi_\alpha'''(u_\alpha)|}{3!} & \text{for} \quad \alpha \in (\frac{1}{3}, \alpha^*] \\ \Delta_\alpha(0) & \text{for} \quad \alpha \in (\alpha^*, \infty) \ . \end{cases}$$

Then

$$|\Delta_\alpha|(u) \le c_\alpha^{(0)} \cdot |1-u|^3 \quad \forall \, u \in [0,1).$$

Remark 8 Let the function c be defined by

$$c(\alpha) = \Delta_\alpha(0), \ \alpha \in (0,\infty)$$

and let $\alpha_c \simeq 0.2779105$ and $\alpha = 1$ be the roots of the numerator of c then

$$c(\alpha) \ge 0 \qquad (6_\alpha)$$

holds true if and only if $\alpha \in [\alpha_c, \infty)$. (Note that $\alpha = 1$ is also a removeable singularity of c.)

Then for every element α of the interval $[\alpha_c, \infty)$ it holds $\Delta_\alpha(u) \ge 0 \ \forall \, u \in [0,1]$.

Cases $\alpha \in (-\infty, 0), \ \alpha = -k, \ k \in (0,\infty)$:
Interval $[1,\infty)$: ($\varphi_{-k}(0) = \frac{1}{(k+1)2^{k+1}}, \ \frac{\varphi''_{-k}(1)}{2!} = \frac{1}{k\cdot 2^{k+3}}, \ \frac{|\varphi'''_{-k}(1)|}{3!} = \frac{1}{k\cdot 2^{k+4}}$)

$$\begin{aligned} \Delta_{-k}(u) &= \varphi_{-k}(u) - \frac{\varphi_{-k}(1)}{2!} \cdot (u-1)^2 \\ &= \frac{1}{1+k}\left(\frac{1+u}{2^{k+1}} - \frac{u}{(1+u^{1/k})^k}\right) - \frac{1}{k \cdot 2^{k+3}} \cdot (u-1)^2 \end{aligned}$$

and hence

$$\Delta_{-k}(0) = \frac{3k-1}{k \cdot (k+1) \cdot 2^{k+3}} \ .$$

Let

$$c_{-k}^{(1)} = \begin{cases} \frac{\left|\varphi'''_{-k}(u_{-k})\right|}{3!} & \text{for} \quad k \in (0, \frac{2}{5}) \\ \frac{\left|\varphi'''_{-k}(1)\right|}{3!} & \text{for} \quad k \in [\frac{2}{5}, \infty) \end{cases}$$

with

$$\varphi'''_{-k}(u) = -\frac{1}{k^2} \frac{u^{1/k-2}}{(1+u^{1/k})^{k+3}} \cdot (k-1+(1+2k)\, u^{1/k}) \ ,$$

$$u_{-k} = (\frac{1-k}{2(1+2k)(1+3k)}(7k+4+\sqrt{\frac{(2+k)(6+11k-k^2)}{1-k}}))^k \ , \ k \in (0,1) \qquad (5_{-k})$$

and $\lim_{k \searrow 0} u_{-k} = 1, \ \lim_{k \nearrow 1} u_{-k} = 0$. Then

$$\left|\Delta_{-k}\right|(u) \le c_{-k}^{(1)} \cdot |u-1|^3 \quad \forall\, u \in [1, \infty) \ .$$

Interval $[0,1]$: Let

$$c_{-k}^{(0)} = \begin{cases} \frac{\left|\varphi'''_{-k}(1)\right|}{3!} & \text{for} \quad k \in [k_{\min}, \frac{2}{5}] \\ \frac{\left|\varphi'''_{-k}(u_{-k})\right|}{3!} & \text{for} \quad k \in (\frac{2}{5}, 1) \\ \Delta_{-k}(0) & \text{for} \quad k \in [1, \infty) \ . \end{cases}$$

Then

$$\left|\Delta_{-k}\right|(u) \le c_{-k}^{(0)} \cdot |1-u|^3 \quad \forall\, u \in [0,1).$$

Remark 9
(i) Note that $\Delta_{-k}(0) \ge 0$ is satisfied if and only if $k \ge \frac{1}{3}$. The latter implies $\Delta_{-k}(u) \ge 0 \ \ \forall\, u \in [0,1]$.
(ii) Condition $\varphi''''_{-k}(1) = 0$ is satisfied for $k = \frac{2}{5}$.

Remark 10 The Application of Theorem 2 to the Special Cases treated in Proposition 2 yield for
Case $\alpha = -1: \ c_{-1} = \max(c_{-1}^{(1)}, c_{-1}^{(0)}) = (\frac{1}{32}, \frac{1}{16}) = \frac{1}{16}$ and
Case $\alpha = 2: \ c_2 = \max(c_2^{(1)}, c_2^{(0)}) = (\frac{1}{8}, \frac{3}{4}) = \frac{3}{4}$,
which are in accordance with our ad hoc solutions.

4. Fisher's Information

Let $(\Omega, \mathscr{A})$ be a nondegenerate measurable space (i.e. $|\mathscr{A}| > 2$ and hence $|\Omega| > 1$), μ a σ-finite measure on $(\Omega, \mathscr{A})$ and $P_\theta, \ \theta \in \Theta$ a family of probability distributions identified by a real valued parameter θ (hence Θ is an open interval of the real line $\mathbb{R}$) given in terms of densities $f_\theta = \frac{dP_\theta}{d\mu}$ with respect to μ . Furthermore, let the support $\Omega_+ = \{x \in \Omega : f_\theta(x) > 0\}$ be independent of the parameter $\theta \in \Theta$ and let

$$I(\theta) = \int_{\Omega_+} (\frac{\partial}{\partial \theta} \ln f_\theta(x))^2 dP_\theta(x)$$

be the corresponding Fisher information measure.
Here and in the sequel we assume that all derivatives exist and are finite.

The following result generalizes the correponding result

$$\frac{\partial^2}{\partial\theta^2} I(P_\theta \| P_{\theta_0}) \,|_{\theta=\theta_0} = I(\theta_0)$$

for the Kulback-Leibler- or I-divergence (the f-divergence for the convex function $\varphi(u) = 1 - u + u\ln u$, cf. also our special case $\alpha = 1$), which may be considered folklore — presumably initiated by Jeffreys (1946) and Kullback and Leibler (1951) — and the results for our special cases for $\alpha = 2$ and $\alpha = -1$ which have also been studied in the related literature; the latter in Vincze (1981).

Theorem 3 Let φ_α, $\alpha \in \mathbb{R}$, be the class of convex functions given above and let $\int_{\Omega_+} \left| \frac{\partial}{\partial\theta} \ln f_\theta(x) \right|^3 dP_\theta(x) < \infty$ be satisfied for all $\theta \in \Theta$. Then for all $\alpha \in (-\infty, -k_{\min}] \cup [\alpha_{\min}, \infty)$ it holds

$$\frac{\partial^2}{\partial\theta^2} I_{\varphi_\alpha}(P_\theta, P_{\theta_0}) \,|_{\theta=\theta_0} = \varphi''_\alpha(1) \cdot I(\theta_0) \ , \quad \theta_0 \in \Theta \ .$$

Proof. Owing to Lemma 1 stating $|\Delta_\alpha(u)| \le c_\alpha \cdot |u-1|^3 \ \forall\, \alpha \in (-\infty, -k_{\min}] \cup [\alpha_{\min}, \infty)$ with $c_\alpha \in (0, \infty)$ it holds

$$\left| \frac{I_{\varphi_\alpha}(P_{\theta_0+\delta}, P_{\theta_0})}{\delta^2} - \frac{\varphi''_\alpha(1)}{2} \cdot \frac{\chi^2(P_{\theta_0+\delta}, P_{\theta_0})}{\delta^2} \right| \le c_\alpha \cdot \frac{\chi^3(P_{\theta_0+\delta}, P_{\theta_0})}{|\delta|^3} \cdot \delta \quad \forall\ \theta_0 + \delta \in \Theta.$$

Application of

$$\lim_{\delta\to 0} \frac{I_{\varphi_\alpha}(P_{\theta_0+\delta}, P_{\theta_0})}{\delta^2} = \frac{1}{2} \frac{\partial^2}{\partial\theta^2} I_{\varphi_\alpha}(P_\theta, P_{\theta_0}) \,|_{\theta=\theta_0}$$

and

$$\lim_{\delta\to 0} \frac{\chi^2(P_{\theta_0+\delta}, P_{\theta_0})}{\delta^2} = I(\theta_0) \quad \text{and} \quad \lim_{\delta\to 0} \frac{\chi^3(P_{\theta_0+\delta}, P_{\theta_0})}{|\delta|^3} = \int_{\Omega_+} \left| \frac{\partial}{\partial\theta} \ln f_\theta(x) \right|^3 dP_\theta(x) \,|_{\theta=\theta_0}$$

gives the result.

■

Appendix

Let $\varphi_\alpha(u)$ as given in (2) for $\alpha \ne 0$, (i.e.

$$\varphi_\alpha(u) = \begin{cases} \frac{sgn(\alpha)}{1-\alpha}((1+u^{1/\alpha})^\alpha - 2^{\alpha-1} \cdot (1+u)) & \text{for} \quad \alpha \in \mathbb{R}\backslash\{0,1\} \\ (1+u)\ln 2 + u\ln u - (1+u)\ln(1+u) & \text{for} \quad \alpha = 1 \end{cases} \ )$$

and hence

$$\varphi_\alpha(0) = \begin{cases} \frac{sgn(\alpha)(1-2^{\alpha-1})}{1-\alpha} & \text{for} \quad \alpha \in \mathbb{R}\backslash\{0,1\} \\ \ln 2 & \text{for} \quad \alpha = 1\ . \end{cases}$$

First derivative:

$$\varphi'_\alpha(u) = \begin{cases} \frac{sgn(\alpha)}{1-\alpha}((1+u^{1/\alpha})^{\alpha-1} u^{1/\alpha-1} - 2^{\alpha-1}) \nearrow & \text{for} \quad \alpha \in \mathbb{R}\backslash\{0,1\} \\ \ln 2 + \ln u - \ln(1+u) \nearrow & \text{for} \quad \alpha = 1 \end{cases}$$

Second derivative:

$$\varphi''_\alpha(u) = \frac{sgn(\alpha)}{\alpha}(1+u^{1/\alpha})^{\alpha-2}u^{1/\alpha-2} > 0 \quad \Rightarrow \quad \frac{\varphi''_\alpha(1)}{2} = \frac{sgn(\alpha)\ 2^{\alpha-3}}{\alpha}$$

Third derivative:

$$\varphi'''_\alpha(u) = \frac{sgn(\alpha)}{\alpha^2}(1+u^{1/\alpha})^{\alpha-3}u^{1/\alpha-3}\cdot(1-2\alpha-(1+\alpha)u^{1/\alpha})$$

Consequences: It holds

$$\varphi'''_\alpha(u) < 0 \begin{cases} \forall\, u \in (0,\infty] \text{ and } \forall\, \alpha \in [\frac{1}{2},\infty) & \text{and} \\ \forall\, u \in [1,\infty] \text{ and } \forall\, \alpha \in (0,\infty). \end{cases}$$

Fourth derivative:

$$\varphi''''_\alpha(u) = \frac{sgn(\alpha)}{\alpha^3}(1+u^{1/\alpha})^{\alpha-4}u^{1/\alpha-4}\cdot((\alpha+1)(7\alpha-4+(2\alpha+1)\,u^{1/\alpha})u^{1/\alpha}+(3\alpha-1)\,(2\alpha-1))\,.$$

For the **Case** $\alpha \in (-\infty,0)$ it turns out appropriate to write $\alpha = -k,\ k \in (0,\infty)$, so that

$$\varphi_{-k}(u) = \frac{1}{1+k}(\frac{1+u}{2^{k+1}} - \frac{u}{(1+u^{1/k})^k}) \implies \varphi_{-k}(0) = \frac{1}{(1+k)\cdot 2^{k+1}}.$$

First derivative:

$$\varphi'_{-k}(u) = \frac{1}{1+k}\left(\frac{1}{2^{k+1}} - \frac{1}{(1+u^{1/k})^{k+1}}\right) \nearrow$$

Second derivative:

$$\varphi''_{-k}(u) = \frac{1}{k}\frac{u^{1/k-1}}{(1+u^{1/k})^{k+2}} > 0 \quad \Rightarrow \quad \frac{\varphi''_{-k}(1)}{2} = \frac{1}{k\cdot 2^{k+3}}$$

Third derivative:

$$\varphi'''_{-k}(u) = \frac{1}{k^2}\frac{u^{1/k-2}}{(1+u^{1/k})^{k+3}}\cdot(1-k-(1+2k)\,u^{1/k})$$

Concequences: It holds

$$\varphi'''_{-k}(u) < 0 \begin{cases} \forall\, u \in (0,\infty] \text{ and } \forall\, k \in [1,\infty) & \text{and} \\ \forall\, u \in [1,\infty] \text{ and } \forall\, k \in (0,\infty). \end{cases}$$

Fourth derivative:

$$\varphi''''_{-k}(u) = \frac{1}{k^3}\frac{u^{1/k-3}}{(1+u^{1/k})^{k+4}}\cdot((1+3k)\,(1+2k)\,u^{2/k}+(k-1)((7k+4)\,u^{1/k}+2k-1)).$$

Investigation of the crucial factors of φ''''_α and φ''''_{-k}

In order to locate the positions u_α and u_{-k} which maximize $|\varphi'''_\alpha(u)|$ and $\left|\varphi'''_{-k}(u)\right|$ we need to investigate the fourth derivative of the functions φ_α and φ_{-k} respectively. For this purpose we consider their crucial factors h_α and h_{-k}.

Let

$$h_\alpha(x) = (\alpha+1)((2\alpha+1)x^2 + (7\alpha-4)x) + (3\alpha-1)(2\alpha-1),\ x \in \mathbb{R},$$

and hence

$$h_\alpha(0) = (3\alpha-1)(2\alpha-1) \begin{cases} <0 & \text{for} \quad \alpha \in (\frac{1}{3}, \frac{1}{2}) \\ =0 & \text{for} \quad \alpha \in \{\frac{1}{3}, \frac{1}{2}\} \\ >0 & \text{for} \quad \alpha \in [0, \frac{1}{3}) \cup (\frac{1}{2}, \infty)\ . \end{cases} \tag{7_α}$$

and

$$h_\alpha(1) = (3\alpha-1)(5\alpha+2) = 0 \quad \text{for} \quad \alpha = \frac{1}{3}\ .$$

Obviously,

$$\text{all functions } h_\alpha \text{ are quadratic and convex polynomials.} \tag{8_α}$$

Furthermore, let $D(\alpha) = \frac{(2-\alpha)(6-11\alpha-\alpha^2)}{1+\alpha}$ and let

$$x_{+,-}(\alpha) = \frac{1}{2(2\alpha+1)}(4-7\alpha \pm \sqrt{D(\alpha)})$$

be the roots of the quadratic equation $h_\alpha(x) = 0$.

Proposition 3$_\alpha$ Let $(\alpha, x) \in D_0 := [0,\infty) \times (0,\infty)$ [3],

$$\hat{x}_-(\alpha) = \max\{0, x_-(\alpha)\},$$

furthermore, let $\alpha^* = \frac{1}{2}(\sqrt{145} - 11)$ and

$$E = \{(\alpha, x) \in D_0 : \hat{x}_-(\alpha) \le x \le x_+(\alpha),\ \alpha \in [0, \alpha^*)\},$$

and $B(E)$ be the boundary of E within D_0. Then

$$h_\alpha(x) \begin{cases} <0 & \text{for all} \quad (\alpha,x) \in E \backslash B(E) \\ =0 & \text{for all} \quad (\alpha,x) \in B(E) \\ >0 & \text{for all} \quad (\alpha,x) \in D_0 \backslash E\ . \end{cases}$$

Proof. For the proof let us define $h_\alpha(x)$ for $x \in \mathbb{R}$. Then the discriminant

$$D(\alpha) \begin{cases} <0 & \text{for} \quad \alpha \in (\alpha^*, 2) \\ =0 & \text{for} \quad \alpha \in \{\alpha^*, 2\} \\ >0 & \text{for} \quad \alpha \in (0, \alpha^*) \cup (2, \infty) \end{cases}$$

and the location $m(\alpha)$ of the minimum of the function h_α is

$$m(\alpha) = \frac{4-7\alpha}{2(1+2\alpha)} \begin{cases} >0 & \text{for} \quad \alpha < \frac{4}{7} \\ =0 & \text{for} \quad \alpha = \frac{4}{7} \\ <0 & \text{for} \quad \alpha > \frac{4}{7}\ , \end{cases}$$

[3] Because of (8_α) we can restrict ourselves here to D_0.

whereby $m(\alpha^*) = \alpha^*/6$. Because of this, (7_α), (8_α) and $\frac{1}{2} < \alpha^* < \frac{4}{7}$ it holds

$$h_\alpha(x) \geq 0 \; \forall x \in [0,\infty) \quad \text{for all } \alpha \in (\alpha^*,\infty).$$

For $\alpha \in [0,\alpha^*]$ the assertion of Proposition 3_α is clear owing to the definitions of $x_-(\alpha)$, $x_+(\alpha)$ and (8_α).

■

Corollary 1_α For all $u \in [1,\infty)$ it holds

$$\frac{\varphi''_\alpha(1)}{2!} \cdot (u-1)^2 - \varphi_\alpha(u) \leq \left\{ \begin{array}{ll} |\varphi'''_\alpha(1)| & \forall\, \alpha \in [\frac{1}{3},\infty) \\ |\varphi'''_\alpha(u_\alpha)| & \forall\, \alpha \in (0,\frac{1}{3}) \end{array} \right\} \cdot \frac{(u-1)^3}{3!} \,.$$

Proof. For $\alpha \geq \frac{1}{3}$ it holds $\varphi''''_\alpha(u) > 0 \, \forall\, u \in [1,\infty)$. Therefore, owing to Proposition 3_α, the negative-valued function $\varphi'''_\alpha(u)$ is monotone increasing on the whole interval $[1,\infty)$. Hence $\varphi'''_\alpha(u)$ takes its minimum value - and consequently $|\varphi'''_\alpha(u| = -\varphi'''_\alpha(u)$ its maximum value - for $u = 1$.
For $\alpha < \frac{1}{3}$ let $x_+(\alpha)$ be the first root of the quadratic equation $h_\alpha(x) = 0$. Then owing to $x_+(\alpha) = u_\alpha^{1/\alpha}$ the function $\alpha \mapsto u_\alpha = x_+^\alpha(\alpha)$, which is given in (5_α), is - again due to Proposition 3_α - the location of the minimum value of the third derivative $\varphi'''_\alpha(u)$ on the interval $(1,\infty)$.
Now, let

$$l_+^{(1)}(\alpha) = \left\{ \begin{array}{lll} 1 & \text{for} & \alpha \in [\frac{1}{3},\infty) \\ u_\alpha & \text{for} & \alpha \in (0,\frac{1}{3}). \end{array} \right.$$

Then, summing-up, it holds

$$0 > \varphi'''_\alpha(u) \geq \varphi'''_\alpha(l_+^{(1)}(\alpha)) \;\; \forall\, u \in [1,\infty).$$

Let $u \in (1,\infty)$. Then applying the Taylor Expansion

$$\varphi_\alpha(u) = \sum_{k=0}^{n} \frac{\varphi_\alpha^{(k)}(1)}{k!} \cdot (u-1)^k + \int_1^u \frac{(u-t)^n}{n!} \cdot \varphi_\alpha^{(n+1)}(t)dt$$

for $n = 2$ yields owing to $\varphi_\alpha(1) = \varphi'_\alpha(1) = 0$ and $\int_1^u \frac{(u-t)^2}{2!}dt = \frac{(u-1)^3}{3!}$

$$\begin{aligned} \varphi_\alpha(u) &= \frac{\varphi''_\alpha(1)}{2!} \cdot (u-1)^2 + \int_1^u \frac{(u-t)^2}{2!} \cdot \varphi'''_\alpha(t)dt \\ &\geq \frac{\varphi''_\alpha(1)}{2!} \cdot (u-1)^2 + \int_1^u \frac{(u-t)^2}{2!} \cdot \varphi'''_\alpha(l_+^{(1)}(\alpha))dt \\ &= \frac{\varphi''_\alpha(1)}{2!} \cdot (u-1)^2 + \varphi'''_\alpha(l_+^{(1)}(\alpha)) \cdot \frac{(u-1)^3}{3!} \end{aligned}$$

and consequently the assertion.

■

Remark 11 Let $\alpha = 0$. Then $m(0) = 2$ and $x_\pm(0) = 2 \pm \sqrt{3}$.

Corollary 2_α Let $u \in [0,1)$. Then it holds for $\alpha \in (0,\infty)$

$$\varphi_\alpha(u) - \frac{\varphi''_\alpha(1)}{2!} \cdot (1-u)^2 \leq \left\{ \begin{array}{lll} |\varphi'''_\alpha(1)| & \text{for} & \alpha \in (0,\frac{1}{3}] \\ |\varphi'''_\alpha(u_\alpha)| & \text{for} & \alpha \in (\frac{1}{3},\alpha^*) \end{array} \right\} \cdot \frac{(1-u)^3}{3!} \,.$$

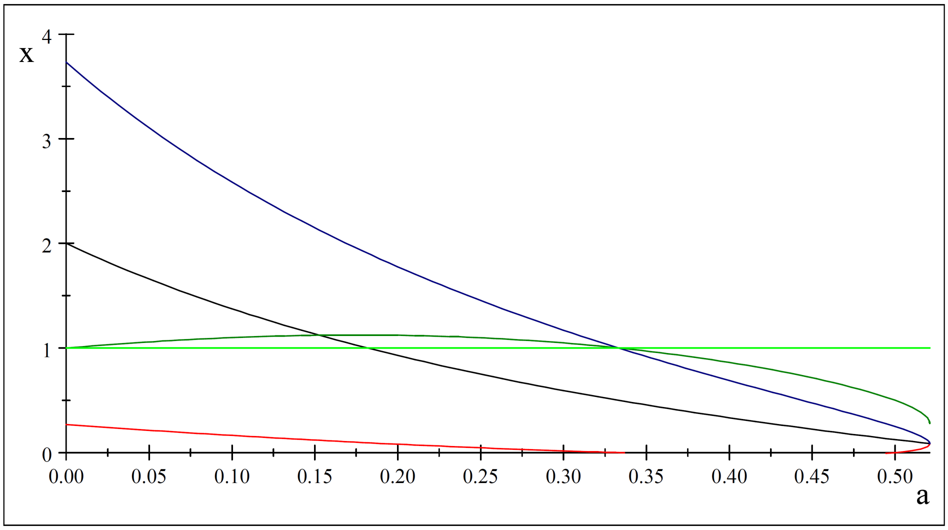

Figure 1: $\alpha \mapsto m(\alpha)$ black, $\alpha \mapsto x_+(\alpha)$ blue, $\alpha \mapsto x_-(\alpha)$ red, $\alpha \mapsto u_\alpha$ green.

Proof. Let

$$l_+^{(0)}(\alpha) = \begin{cases} 1 & \text{for} \quad \alpha \in (0, \frac{1}{3}] \\ u_\alpha & \text{for} \quad \alpha \in (\frac{1}{3}, \alpha^*). \end{cases}$$

Owing to Proposition 3_α it holds

$$\varphi_\alpha''''(u) < 0 \ \text{ for } \ x_-(\alpha) < u \leq 1$$

and hence $\varphi_\alpha'''(u) > \varphi_\alpha'''(l_+^{(0)}(\alpha)) < 0$. This yields owing to $\int_u^1 \frac{(t-u)^2}{2!} dt = \frac{(1-u)^3}{3!}$

$$\begin{aligned} \varphi_\alpha(u) &= \frac{\varphi_\alpha''(1)}{2!} \cdot (u-1)^2 + \int_1^u \frac{(u-t)^2}{2!} \cdot \varphi_\alpha'''(t) dt \\ &= \frac{\varphi_\alpha''(1)}{2!} \cdot (1-u)^2 - \int_u^1 \frac{(t-u)^2}{2!} \cdot \varphi_\alpha'''(t) dt \\ &\leq \frac{\varphi_\alpha''(1)}{2!} \cdot (1-u)^2 - \int_u^1 \frac{(t-u)^2}{2!} \cdot \varphi_\alpha'''(l_+^{(0)}(\alpha)) dt \\ &= \frac{\varphi_\alpha''(1)}{2!} \cdot (1-u)^2 + \left| \varphi_\alpha'''(l_+^{(0)}(\alpha)) \right| \cdot \frac{(1-u)^3}{3!} \end{aligned}$$

and consequently the assertion.

■

In addition, let

$$h_{-k}(x) = (1+3k)(1+2k)x^2 + (k-1)((7k+4)x + 2k - 1)$$

and hence

$$h_{-k}(0) = (k-1)(2k-1) \begin{cases} < 0 & \text{for} \quad k \in (\frac{1}{2}, 1) \\ = 0 & \text{for} \quad k \in \{\frac{1}{2}, 1\} \\ > 0 & \text{for} \quad k \in [0, \frac{1}{2}) \cup (1, \infty) \, . \end{cases} \qquad (7_{-k})$$

and

$$h_{-k}(1) = (5k-2)(1+3k) = 0 \quad \text{for} \quad k = \frac{2}{5}.$$

Again, obviously,

$$\text{all functions } h_{-k} \text{ are quadratic and convex polynomials.} \qquad (8_{-k})$$

Furthermore, let $D(k) = (1-k)(2+k)(6+11k-k^2)$ and let

$$x_{+,-}(k) = \frac{1}{2(1+2k)(1+3k)}((1-k)(7k+4) \pm \sqrt{D(k)})$$

be the the roots of the quadratic equation $h_{-k}(x) = 0$.

Proposition 3$_{-k}$ Let $(k,x) \in D_0 = [0,\infty) \times (0,\infty)$ [4],

$$\hat{x}_-(k) = \max\{0, x_-(k)\},$$

furthermore, let

$$E = \{(k,x) \in D_0 : \hat{x}_-(k) \leq x \leq x_+(k),\ k \in [0,1)\},$$

and $B(E)$ be the boundary of E within D_0. Then

$$h_{-k}(x) \begin{cases} < 0 & \text{for all} \quad (k,x) \in E \backslash B(E) \\ = 0 & \text{for all} \quad (k,x) \in B(E) \\ > 0 & \text{for all} \quad (k,x) \in D_0 \backslash E \ . \end{cases}$$

Proof. For the proof let us, again, define $h_{-k}(x)$ for $x \in \mathbb{R}$. Let $k^* = \frac{1}{2}(\sqrt{145}+11)$. Then the discriminant

$$D(k) \begin{cases} < 0 & \text{for} \quad k \in (1,k^*) \\ = 0 & \text{for} \quad k \in \{1,k^*\} \\ > 0 & \text{for} \quad k \in (0,1) \cup (k^*,\infty) \end{cases}$$

and the location $m(k)$ of the minimum of the function h_{-k} is

$$m(k) = \frac{(1-k)(7k+4)}{2(1+2k)(1+3k)} \begin{cases} > 0 & \text{for} \quad k < 1 \\ = 0 & \text{for} \quad k = 1 \\ < 0 & \text{for} \quad k > 1 \ . \end{cases}$$

Because of this, (7_{-k}) and (8_{-k}) it holds

$$h_{-k}(x) \geq 0 \;\forall\, x \in [0,\infty) \quad \text{for all} \quad k \in (1,\infty) \ .$$

For $k \in [0,1]$ the assertion of Proposition 3_{-k} is clear owing to the definitions of $x_-(k)$, $x_+(k)$ and (8_{-k}). ■

Similarly to Corollary 1_α one obtains from Proposition 3_{-k}

[4] Because of (8_{-k}) we can restrict ourselves here to D_0.

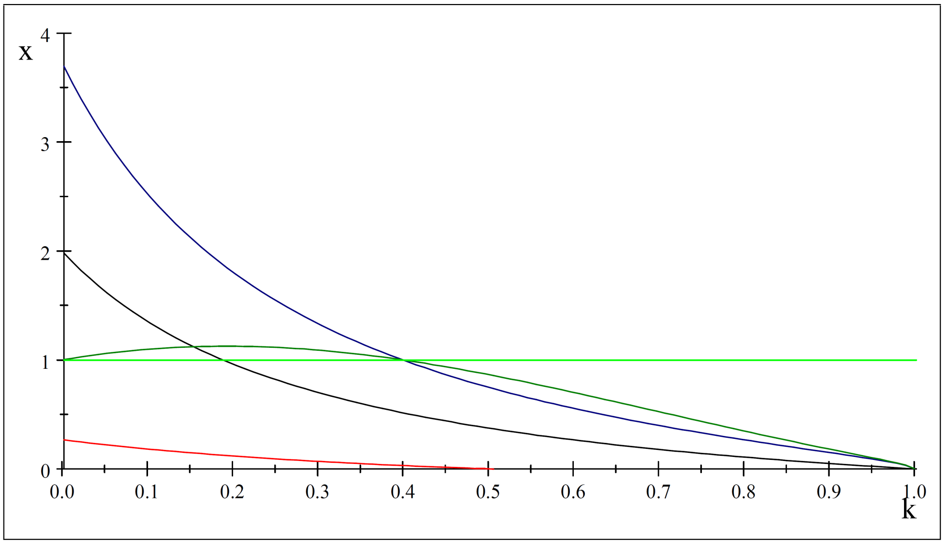

Figure 2: $k \mapsto m(k)$ black, $k \mapsto x_+(k)$ blue, $k \mapsto x_-(k)$ red, $k \longmapsto u_{-k}$ green.

Corollary 1$_{-k}$ For all $u \in [1,\infty)$ it holds

$$\frac{\varphi''_{-k}(1)}{2!} \cdot (u-1)^2 - \varphi_{-k}(u) \leq \left\{ \begin{array}{ll} \left|\varphi'''_{-k}(1)\right| & \forall\, k \in [\frac{2}{5},\infty) \\ \left|\varphi'''_{-k}(u_{-k})\right| & \forall\, k \in (0,\frac{2}{5}) \end{array} \right\} \cdot \frac{(u-1)^3}{3!} .$$

Corollary 2$_{-k}$ Let $u \in [0,1)$. Then it holds for $k \in (0,\infty)$

$$\varphi_{-k}(u) - \frac{\varphi''_{-k}(1)}{2!} \cdot (1-u)^2 \leq \left\{ \begin{array}{lll} \left|\varphi'''_{-k}(1)\right| & \text{for} & \forall\, k \in (0,\frac{2}{5}] \\ \left|\varphi'''_{\alpha}(u_\alpha)\right| & \text{for} & \forall\, k \in (\frac{2}{5},1) \end{array} \right\} \cdot \frac{(1-u)^3}{3!} .$$

Proposition 4$_\alpha$ Let $\alpha_c \simeq 0.27779105$ and $\alpha = 1$ be the roots of the numerator of

$$c(\alpha) = \Delta_\alpha(0) = \frac{8\alpha - 2^\alpha(1+3\alpha)}{8\alpha(1-\alpha)}.$$

Then for all $\alpha \in [\alpha_c,\infty)$ it holds

$$\Delta_\alpha(u) \geq 0 \;\; \forall\, u \in [0,1),$$

with strict inequality for all cases except for the case $\Delta_{\alpha_c}(0) = 0$.
Proof. Case $\alpha \in [\frac{1}{2},\infty)$: Owing to the definition of the function

$$u \mapsto \Delta_\alpha(u) = \varphi_\alpha(u) - \frac{\varphi''_\alpha(1)}{2!} \cdot (u-1)^2$$

in Lemma 1, Section 3, it holds

$$\Delta_\alpha(1) \; (= \varphi_\alpha(1)) = 0 \,. \tag{9_α}$$

Furthermore, because of $\Delta'_\alpha(u) = \varphi'_\alpha(u) - \varphi''_\alpha(1)(u-1)$ it is

$$\Delta'_\alpha(1) \; (= \varphi'_\alpha(1)) = 0 \; , \tag{10_α}$$

because of $\Delta''_\alpha(u) = \varphi''_\alpha(u) - \varphi''_\alpha(1)$

$$\Delta''_\alpha(1) = 0 \tag{11_α}$$

and, owing to $\alpha \geq \frac{1}{2}$, finally

$$\Delta'''_\alpha(u) = \varphi'''_\alpha(u) = \frac{1}{\alpha^2}(1+u^{1/\alpha})^{\alpha-3}u^{1/\alpha-3} \cdot (1-2\alpha-(1+\alpha)u^{1/\alpha}) < 0 \; \forall \, u \in [0,\infty).$$

Therefore $\Delta''_\alpha(u)$ is monotone decreasing and due to (11_α)

$$\Delta''_\alpha(u) > 0 \; \forall \, u \in [0,1).$$

Therefore $\Delta'_\alpha(u)$ is monotone increasing and due to (10_α)

$$\Delta'_\alpha(u) < 0 \; \forall \, u \in [0,1).$$

Consequently $\Delta_\alpha(u)$ is monotone decreasing and due to (9_α), finally,

$$\Delta_\alpha(u) > 0 \; \forall \, u \in [0,1).$$

■

Proof. Case $\alpha \in [\alpha_c, \frac{1}{2})$: The proof of this case is a little more complicated than the above one and needs, in view of $\Delta'_\alpha(0) = 0$ for $\alpha = \frac{1}{3}$, the distinction of the subcases $\alpha \in [\frac{1}{3}, \frac{1}{2})$ and $\alpha \in [\alpha_c, \frac{1}{3})$. In order to shorten our paper we skip this part. Of course, the proof is available from the authors upon request.

■

Very much along the lines of the above proof one can also show the corresponding result for the case $\alpha = -k, \; k \in (0,\infty)$. In this it is appropriate to distinguish the cases $k \in [1,\infty)$ and $k \in [\frac{1}{3}, 1)$ and the subcases $k \in [k^{**}, 1)$ and $k \in [\frac{1}{3}, k^{**})$, whereby $k = k^{**} \simeq 0.42313$ is the solution of the equation $\Delta'_{-k}(0) = 0$ with

$$\Delta'_{-k}(0) = \frac{1+k(3-2^{k+2})}{k(k+1)2^{k+2}}.$$

Proposition 4_{-k} For all $k \in [\frac{1}{3}, \infty)$ it holds

$$\Delta_{-k}(u) \geq 0 \;\; \forall \, u \in [0,1)$$

with strict inequality for all cases except for the case $\Delta_{-1/3}(0) = 0$.

Remark 12_{-k} It turns out that

$$\delta'_{-k}(0) = \frac{1+k\cdot(2^{k+3}-15)}{k(k+1)2^{k+3}}.$$

Let, furthermore, $k_+ \simeq 0.7775715$ be the larger of the two solutions of $\delta'_{-k}(0) = 0$. Then $\delta'_{-k}(0) > 0$ for $k \in (k_+, \infty)$.

Proposition 5$_{-k}$ For all $k \in [1, \infty)$ it holds

$$\Delta_{-k}(u) \leq \Delta_{-k}(0) \cdot (1-u)^3 \quad \forall\, u \in [0,1].$$

Proof. Let

$$\delta_{-k}(u) = \Delta_{-k}(0) \cdot (1-u)^3 - \Delta_{-k}(u).$$

Then, consequently,

$$\delta_{-k}(0) = 0 \text{ and } \delta_{-k}(1)\ (= \Delta_{-k}(1)) = 0 \tag{9_{-k}}$$

and because of $\delta'_{-k}(u) = -3 \cdot \Delta_{-k}(0) \cdot (1-u)^2 - \Delta'_{-k}(u)$

$$\delta'_{-k}(1)\ (= -\Delta'_{-k}(1)) = 0 \tag{$10^{(1)}_{-k}$}$$

and in view of Remark 12_{-k}

$$\delta'_{-k}(0) > 0. \tag{$10^{(0)}_{-k}$}$$

Furthermore, $\delta''_{-k}(u) = 6 \cdot \Delta_{-k}(0) \cdot (1-u) - \Delta''_{-k}(u)$ and

$$\delta''_{-k}(0) = \begin{cases} -\frac{1}{2} & \text{for} \quad k = 1 \\ -\infty & \text{for} \quad k \in (1,\infty) \end{cases} \quad \text{and} \quad \delta''_{-k}(1)\ (= -\Delta''_{-k}(1)) = 0 \tag{11_{-k}}$$

and $\delta'''_{-k}(u) = -6 \cdot \Delta_{-k}(0) - \Delta'''_{-k}(u)$ and hence

$$\delta'''_{-k}(1) = -6 \cdot \Delta_{-k}(0) - \Delta'''_{-k}(1) = 6 \cdot \Big(\frac{|\varphi'''_{-k}(1)|}{3!} - \Delta_{-k}(0)\Big) < 0, \tag{12_{-k}}$$

the latter in view of

$$\Delta_{-k}(0) - \frac{|\varphi'''_{-k}(1)|}{3!} = \frac{3k-1}{k \cdot (k+1)2^{k+3}} - \frac{1}{k \cdot 2^{k+4}} = \frac{5k-3}{k \cdot (k+1)2^{k+4}} > 0.$$

Owing to Proposition 3_{-k} it holds $\varphi''''_{-k}(u) > 0\ \forall\ u \in [0,\infty)$ for all $k \geq 1$. Consequently, the function Δ'''_{-k} is strictly monotone increasing and hence δ'''_{-k} is strictly monotone decreasing. Therefore, because of

$$\delta'''_{-k}(0) = \begin{cases} \frac{21}{8} & \text{for} \quad k = 1 \\ \infty & \text{for} \quad k \in (1,\infty) \end{cases}$$

and (12_{-k}) there exists a unique value $u_3 \in (0,1)$ satisfying $\delta'''_{-k}(u_3) = 0$ and it holds

$$\delta'''_{-k}(u) \begin{cases} > 0 & \text{for all} \quad u \in [0, u_3) \\ < 0 & \text{for all} \quad u \in (u_3, 1]\,. \end{cases}$$

Consequently the function

$$\delta''_{-k} \text{ is strictly monotone} \begin{cases} \text{increasing on the interval} & [0, u_3) \\ \text{decreasing on the interval} & (u_3, 1]\,. \end{cases}$$

Owing to (11_{-k}) the maximum value of δ''_{-k} is $\delta''_{-k}(u_3) > 0$ and there exists a unique value $u_2 \in (0,u_3)$ satisfying $\delta''_{-k}(u_2) = 0$ and it holds

$$\delta''_{-k}(u) \begin{cases} < 0 & \text{for all} \quad u \in [0,u_2) \\ > 0 & \text{for all} \quad u \in (u_2,1] \, . \end{cases}$$

Consequently the function

$$\delta'_{-k} \text{ is strictly monotone} \begin{cases} \text{decreasing on the interval} & [0,u_2] \\ \text{increasing on the interval} & [u_2,1] \, . \end{cases}$$

Due to ($10^{(0)}_{-k}$) and ($10^{(1)}_{-k}$) the minimum value of δ'_{-k} is $\delta'_{-k}(u_2) < 0$ and there exists a unique value $u_1 \in (0,u_2)$ satisfying $\delta'_{-k}(u_1) = 0$ and it holds

$$\delta'_{-k}(u) \begin{cases} > 0 & \text{for all} \quad u \in [0,u_1) \\ < 0 & \text{for all} \quad u \in (u_1,1] \, . \end{cases}$$

Consequently the function

$$\delta_{-k} \text{ is strictly monotone} \begin{cases} \text{decreasing on the interval} & [0,u_1] \\ \text{increasing on the interval} & [u_1,1] \, . \end{cases}$$

(9_{-k}), finally, yields the result.

■

The proof of the corresponding result for the class $\alpha \in (0,\infty)$ is similar to that of Proposition 5_{-k} and the relevant Remark is

Remark 12_α It turns out that

$$\delta'_\alpha(0) = \begin{cases} \frac{2^\alpha \cdot (1+15\alpha) - 24\alpha}{8\alpha(1-\alpha)} & \text{for} \quad \alpha \in (0,1) \\ \infty & \text{for} \quad \alpha \in [1,\infty). \end{cases}$$

Let, furthermore, $\alpha_+ \simeq 0.496066$ be the larger one of the two solutions of $\delta'_\alpha(0) = 0$ for $\alpha \in (0,1)$. Then $\delta'_\alpha(0) > 0$ for $\alpha \in (\alpha_+,\infty)$.

Proposition 5_α For all $\alpha \in (\alpha^*,\infty)$ it holds

$$\Delta_\alpha(u) \leq \Delta_\alpha(0) \cdot (1-u)^3 \quad \forall\, u \in [0,1].$$

For all cases $k \in (0,\frac{1}{3})$ there exits a value $u_0(k) \in (0,1)$ for which $\Delta_{-k}(u_0(k)) = 0$ and

$$\Delta_{-k}(u) \begin{cases} < 0 & \forall\, u \in [0,u_0(k)) \\ > 0 & \forall\, u \in (u_0(k),1) \end{cases}$$

and for all $\alpha \in (0,\alpha_c]$ there exits a corresponding value $u_0(\alpha) \in (0,1)$ concerning Δ_α instead of Δ_{-k}. In this respect the validity of the Propositions 5_{-k} and 5_α, respectively, can be extended to certain lower bounds $k_{\min} \in (0,\frac{1}{3})$ and $\alpha_{\min} \in (0,\alpha_c)$, however not beyond these. In order not to extend the length of our paper unduly we also state these results without proof.

Proposition 6_{-k} Let for $k \in (0, \frac{1}{3})$

$$\delta_{-k}(u) = \frac{\left|\varphi'''_{-k}(1)\right|}{3!} \cdot (1-u)^3 - \left|\Delta_{-k}\right|(u) \ .$$

Then

$$\delta_{-k}(0) = \frac{7k-1}{k(k+1)\cdot 2^{k+4}} > 0 \ \text{für} \ k > \frac{1}{7}$$

and the function $k \mapsto \delta_{-k}(0)$ is monotone increasing on the interval $(0, \frac{1}{3})$ and it holds $\delta_{-k}(0) \in [0, 3\cdot 2^{-\frac{13}{3}}] \ \forall \, k \in [\frac{1}{7}, \frac{1}{3}]$ where $\delta_{-1/7}(0) = 0$.
Furthermore, let $k_{\min} \in (\frac{1}{7}, \frac{1}{3})$ be

$$k_{\min} = \sup\{k \in [\frac{1}{7}, \frac{1}{3}] : \exists \, u \in [0,1) : \delta_{-k}(u) \leq 0\} \ .$$

Then the lower bound is $k_{\min} \simeq 0.14539$ and, in addition to Proposition 5_{-k}, it clearly holds for all $k \in [k_{\min}, \frac{1}{3})$

$$\left|\Delta_{-k}\right|(u) \leq \frac{\left|\varphi'''_{-k}(1)\right|}{3!} \cdot (1-u)^3 \ \ \forall \, u \in (0,1) \ .$$

Proposition 6_{α} Let for $\alpha \in (0, \alpha_c)$

$$\delta_{\alpha}(u) = \frac{|\varphi'''_{\alpha}(1)|}{3!} \cdot (1-u)^3 - |\Delta|_{\alpha}(u) \ .$$

Then

$$\delta_{\alpha}(0) = \frac{16\alpha - 2^{\alpha}(1+7\alpha)}{16\alpha(1-\alpha)} > 0 \text{ for } \alpha > \alpha_0 \simeq 0.13147$$

and the function $\alpha \mapsto \delta_{\alpha}(0)$ is monotone increasing on the interval $(0,1)$ and it holds $\delta_{\alpha}(0) \in [0, 0.27255] \ \forall \, \alpha \in [\alpha_0, \alpha_c]$ where $\delta_{\alpha_0}(0) = 0$.
Furthermore, let $\alpha_{\min} \in (\alpha_0, \alpha_c)$ be

$$\alpha_{\min} = \sup\{\alpha \in [\alpha_0, \alpha_c] : \exists \, u \in [0,1) : \delta_{\alpha}(u) \leq 0\} \ .$$

Then the lower bound is $\alpha_{\min} \simeq 0.1337$ and, in addition to Proposition 5_{α} it clearly holds for all $\alpha \in [\alpha_{\min}, \alpha_c)$

$$|\Delta_{\alpha}|(u) \leq \frac{|\varphi'''_{\alpha}(1)|}{3!} \cdot (1-u)^3 \ \ \forall \, u \in (0,1) \ .$$

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