MULTI PERSON DECISION MAKING – A COMPROMISE SOLUTION

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Abstract: In this paper a group of l decision makers who are confronted by the problem of choosing a mutually acceptable solution to a statistical decision problem, is considered. If consensus is not reached, a solution through compromise is called for. A measure of similarity or agreement between each pair of decision makers is defined, where this measure can be used to assign weights to the decision makers. These weights give an indication of the importance of the decision makers in terms of relative agreement with others in the group. A solution through compromise can be found by using these weights in the calculation of a randomised decision.

1. Introduction

One of the primary functions of groups is to make decisions. This however can be easier said than done. Classically, consensus is defined as the full and unanimous agreement of all the decision makers (*DMs*) regarding all the possible alternatives (Herrera-Viedma, Herrera and Chiclana, 2002). The situation of *DMs*, confronted by the problem to reach consensus or, if not possible, to choose a mutually acceptable solution to a decision problem, is considered.

DMs have long relied on expert judgement to inform their decision making. Three broad contexts can be considered (French, 2011): Firstly, the expert problem where a group of experts are asked for advice by a *DM* who has the responsibility for the consequences of the decision. Secondly, the group decision problem where the group itself is jointly responsible and accountable for the decision. In this case the *DMs* may wish to combine their judgements in some formal structured way. Lastly, the textbook problem where it may be required from the group to give their judgements for others to use in the future in as yet undefined circumstances.

According to French (2011), it is likely that a combination of the contexts will occur and that a group of *DMs* might be informed by a group of experts before making their decision. French (2011) views this as a two stage problem in which each *DM* listens to the experts and updates his/her probabilities in the light of their opinions and then the *DMs* act as a group coming to a decision. This context is considered in this paper and a method to combine the *DM*'s judgements is suggested.

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In Section 2 an overview of the elicitation and combination of expert opinion is given and Section 3 deals with the definition of a measure of similarity and the calculation of weights assigned to the *DMs*. The general case of more than two outcomes is considered in Section 4 and Section 5 focuses on utilising these weights in a compromise solution if consensus is not reached.

2. The Elicitation and Combination of Expert Opinion

As in Garisch (2009), consider the situation of l *DMs*, denoted by $DM_1, DM_2, \cdots, DM_l$, who are confronted by the problem to choose a mutually acceptable solution to a statistical decision problem. If the *DMs* have widely different preference structures, it seems likely that consensus will not be reached and that a solution through compromise is called for. Consider a parameter space with two points, $\Theta = \{\theta_1, \theta_2\}$ and let $P_i(\Theta = \theta_1) = p_i$ denote the prior probability for DM_i that Θ equals θ_1, $i = 1, 2, \cdots, l$. The decision space is denoted by $D = \{d_1, d_2, \cdots, d_n\}$, where the d_j, $j = 1, 2, \cdots, n$, represent the available decisions. Suppose further each *DM* has his own utility function. For DM_i this utility function is given in Table 1.

Table 1: Utility function for DM_i.

		Decisions			
		d_1	d_2	$\cdots$	d_n
Parameter space	θ_1	U_{i11}	U_{i12}	$\cdots$	U_{i1n}
	θ_2	U_{i21}	U_{i22}	$\cdots$	U_{i2n}

The vector of expected utilities for DM_i is denoted by $\mathbf{U_i'} = (U_{i1}, U_{i2}, \cdots, U_{in})$, $i = 1, 2, \cdots, l$, where $U_{ij} = U_{i1j}p_i + U_{i2j}(1 - p_i)$. Each *DM* will rank the decisions according to the expected utilities. The decision with maximum expected utility will be chosen and typically consensus will not be reached.

Suppose information additional to the prior probabilities of the *DMs* can be provided by an expert or a group of experts. According to Cooke and Goosens (2010), the experts may possess valuable knowledge about the parameters in the subject matter, and they can draw from their vast expertise in their particular field of interest to assess unknown quantities. The knowledge they can offer is not certain but the experts can specify "degrees of belief" that can be quantified and aggregated to give optimal choices of parameters.

Various methods of eliciting and combining expert opinion have been developed over the years and Cooke's classical model has been a popular approach for over 20 years. According to Aspinall (2010) the weighting of the opinion of each expert based on their knowledge and ability to judge uncertainties relevant to the problem can produce a "rational consensus". The classical method of Cooke was developed towards this research effort of obtaining a "rational consensus", and it has been used extensively in the field of risk analysis (Ryan, Mazzuchi, Ryan, Lopez De La Cruz and Cooke, 2012). An expert's final weight is then given by the normalised product of a calibration score and an information score. A fundamental assumption of the classical model is that experts' future performance can be judged on the basis of how they have performed in the past.

The Traditional Committee method is a behavioural aggregation method where the experts interact to achieve homogeneity of information on the parameters of interest (Cooke and Goosens, 2010).

The Delphi method is described by Eggstaff, Mazzuchi and Sarkani (2014) as a behavioural technique that is prone to psychological bias that may affect the validity of the results. In the Paired Comparison method, Cooke and Goosens (2010) explain that alternatives are compared and ranked pairwise. According to Mazzuchi, Linzey and Bruning (2008) the Negative Exponential Life (NEL) model is a popular model based on the paired comparison method for expert judgment analysis. In the NEL model n components are compared pairwise and then ranked. It can then be analysed whether each expert is specifying a true preference structure in his/her answers or merely assigning answers randomly.

According to French (2011) only two of the approaches that he explored earlier stood the test of time. One is the Bayesian approach and the other Opinion Pooling. Bayesian approaches treat the expert's judgements as data for the *DM* and then seek to develop appropriate likelihood functions to represent the *DM*'s relative confidence in the experts. Opinion pools simply weight together the expert's judgements using a weighted arithmetic or geometric mean or something more general. Van Noortwijk, Dekker, Cooke and Mazzuchi (1992) initiated the use of expert judgment analysis within the maintenance environment. Their procedure involved the Histogram technique, the Supra Bayes approach and use of the Dirichlet distribution.

According to Herrera-Viedma et al. (2002) each expert has his/her own ideas, attitudes, motivations, and personality, and it is quite natural to consider that different experts will give their preferences in a different way. Thus, before the *DMs* can acquire information from the experts, a uniform representation of the expert's preferences must be obtained. In this paper the assumption is made, as in French (2011) and Garcia and Puig (2004), that the experts are asked to provide probability judgements.

3. A Measure of Similarity

Suppose information additional to the prior probabilities of the *DMs* can be provided in the form of a probability k for the occurrence of the event $\Theta = \theta_1$. This is denoted by

$$P(\text{“}\Theta = \theta_1\text{”}) = k.$$

According to Garisch and Groenewald (1996) it must be decided beforehand whether to use an expert or not, thus whether the information provided by the expert will lead to consensus. It is however not possible to know with certainty if the use of a specific expert will result in consensus, but the probability can be calculated for each *DM*. It is suggested that experts with probabilities for consensus greater than some value β should be consulted.

Suppose the decision d_E is added to the decision space by each *DM* where d_E denotes the decision to consult an expert. The utility function for DM_i is now given in Table 2.

Table 2: Updated utility function for DM_i.

		Decisions				
		d_1	d_2	$\cdots$	d_n	d_E
Parameter space	θ_1	U_{i11}	U_{i12}	$\cdots$	U_{i1n}	U_{i1E}
	θ_2	U_{i21}	U_{i22}	$\cdots$	U_{i2n}	U_{i2E}

If there is no gain if consensus is reached and no cost involved in the consultation of the expert,

$$U_{i1E} = \sum_{j=1}^{n} P\left[\text{choose } d_j \,|\Theta = \theta_1\right] U_{i1j}$$

and

$$U_{i2E} = \sum_{j=1}^{n} P\left[\text{choose } d_j \,|\Theta = \theta_2\right] U_{i2j}.$$

As in Garisch and Groenewald (1996), it is assumed that the distributions of $k\,|\Theta = \theta_i$, $i = 1,2$ are as follows:

$$f\left(k\,|\Theta = \theta_1\right) = \frac{\Gamma(r+t)}{\Gamma(r)\Gamma(t)} k^{r-1}(1-k)^{t-1}$$

and

$$f\left(k\,|\Theta = \theta_2\right) = \frac{\Gamma(r+t)}{\Gamma(r)\Gamma(t)} k^{t-1}(1-k)^{r-1}, \;\; 0 < k < 1, r > 0, t > 0.$$

Thus symmetry of the conditional distributions is assumed and the mean and variance of $k\,|\Theta = \theta_1$ are

$$\mu = \frac{r}{r+t}$$

and

$$\sigma^2 = \frac{rt}{(r+t)^2(r+t+1)},$$

respectively. Should DM_i make use of this information, the posterior probability that $\Theta = \theta_1$ is

$$p_i' = P_i\left(\Theta = \theta_1\,|k\right) = \left[1 + \left(\frac{1-p_i}{p_i}\right)\left(\frac{1-k}{k}\right)^{r-t}\right]^{-1}, \;\; i = 1,2,\cdots,l.$$

Again it is likely that consensus will not be reached and a solution through compromise can be reached by assigning a weight to each DM.

The values of r and t give an indication of the quality or significance of the additional information. If $r = t$ then $\mu = 1/2$ and $p_i' = p_i$, so the additional information has no value. For $r \ggg t$, $\mu \to 1$ and for $r \lll t$, $\mu \to 0$. For small values of σ^2 and values of μ significantly different from

$1/2$, high quality information is provided. It should be noted that, without loss of generality, it is only necessary to consider values of k where $k > 1/2$.

The quality of the information received from the experts can be measured in many ways. Garcia and Puig (2004) assume that in addition to each expert opinion being represented by a probability, it is also associated with a confidence level that expresses the conviction of the expert on its own judgement. According to Zapata-Vázquez, O'Hagan and Bastos (2014), the usual approach to elicit knowledge about a set of uncertain proportions which must sum to 1 is to assume that the expert's knowledge can be represented by a Dirichlet distribution. So far we only considered the one dimensional case of the Dirichlet distribution, namely the beta distribution. The general case is discussed in Section 4.

If it is decided to use an expert, the posterior probabilities are calculated. Suppose d_j is the optimal decision for DM_i if $a_{ji} \leq p_i' \leq b_{ji}$, $j = 1, 2, \cdots, n$, $i = 1, 2, \cdots, l$.

Thus, for a specified value of k, $k > 1/2$ the optimal decision for DM_i is d_j if

$$A_{ji|k} \leq g(\mu, \sigma) \leq B_{ji|k}$$

where

$$\begin{aligned} g(\mu, \sigma) &= (2\mu - 1)\left[\frac{\mu(1-\mu)}{\sigma^2} - 1\right], \\ A_{ji|k} &= \frac{\ln\left[\left(\frac{p_i}{1-p_i}\right)\left(\frac{1-a_{ji}}{a_{ji}}\right)\right]}{\ln\left(\frac{1-k}{k}\right)}, \end{aligned}$$

and

$$B_{ji|k} = \frac{\ln\left[\left(\frac{p_i}{1-p_i}\right)\left(\frac{1-b_{ji}}{b_{ji}}\right)\right]}{\ln\left(\frac{1-k}{k}\right)}, \quad j = 1, 2, \cdots, n, \; i = 1, 2, \cdots, l.$$

Then

$$C_{ji|k} = \left\{(\mu, \sigma) \,\middle|\, A_{ji|k} \leq g(\mu, \sigma) \leq B_{ji|k}\right\}$$

denotes, for a given value of k, the set of all pairs (μ, σ) such that d_j is the optimal decision for DM_i.

For two decision-makers, DM_i and DM_t, $i \neq t$, $C_{ji|k} \cap C_{jt|k}$ is the set of all pairs (μ, σ) such that consensus is reached on d_j for a given value of k, while the set

$$D_{it} = \left\{(k, \mu, \sigma) \,\middle|\, \bigcup_k \left(\bigcup_j \left(C_{ji|k} \cap C_{jt|k}\right)\right)\right\}$$

contains all triples (k, μ, σ) such that consensus is reached between these two DMs.

Let s_{it} denote the similarity between DM_i and DM_t. Using a principle similar to the Jaccard Coefficient, s_{it} can be defined as the number of elements in D_{it} divided by the number of elements in the set D of all possible triples (k, μ, σ).

Thus

$$s_{it} = s_{ti} = \frac{|D_{it}|}{|D|}, \text{ where } 0 \leq s_{it} \leq 1.$$

The Jaccard Similarity Coefficient is a statistic used for comparing the similarity and diversity of sets. A measure of distance or diversity between sets is defined as one minus the similarity coefficient.

These measures of similarity can be used to assign weights to the DMs. The weight assigned to DM_i can be defined as $w_i = \frac{s_{i.}}{\sum_t s_{t.}}$, where $s_{i.} = \sum_{j \neq i} s_{ij}$, $i = 1, 2, \cdots, l$. This weight gives an indication of the importance of DM_i in the group by taking the similarity between DM_i and the rest of the DMs, $DM_1, DM_2, \cdots, DM_{i-1}, DM_{i+1}, \cdots, DM_l$ into account.

It should be noted that, using the similarity measure, various methods are available to calculate weights. The method suggested in this paper should be viewed as one of a number of approaches that can be considered. What makes one better than the other may differ from one situation to the next depending on the experts available, the information that can be obtained from the experts, etc.

Example 1. In this example two DMs are considered and the measure of similarity is calculated. Consider DM_1 with the following utility function given in Table 3.

Table 3: Utility function for DM_1.

		Decisions		
	DM_1	d_1	d_2	d_3
Parameter space	θ_1	5	2	4
	θ_2	1	4	3

For a prior probability of $p_1 = 0.7$, the optimal decision is d_1. Suppose that additional information $P(\text{“}\Theta = \theta_1\text{”}) = k = 0.9$ for the occurrence of the event “$\Theta = \theta_1$” is provided. The posterior probability, and therefore the optimal decision will depend on the values of μ and σ^2.

Figure 1 shows which values of μ and σ^2 will result in decisions d_1, d_2, and d_3 respectively.

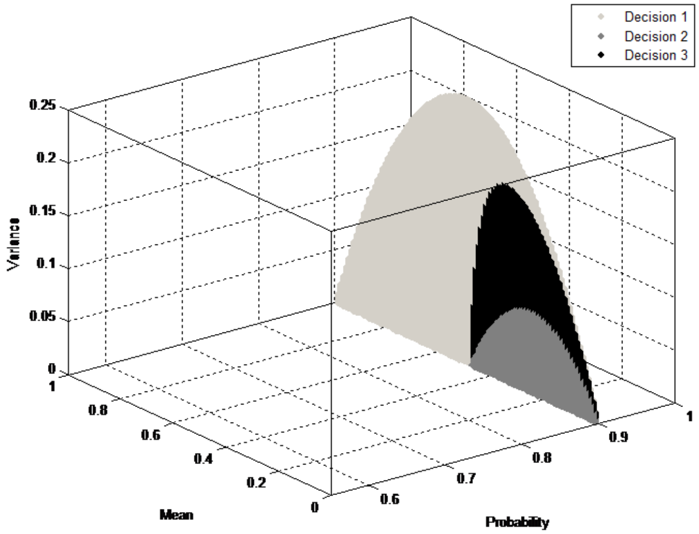

Figure 1: Values of μ and σ^2 that will result in decisions d_1, d_2, and d_3 respectively.

Consider now DM_2 with the following utility function given in Table 4.

Table 4: Utility function for DM_2.

		Decisions		
	DM_2	d_1	d_2	d_3
Parameter space	θ_1	11	6	10
	θ_2	3	5	4

Suppose this second DM has a prior probability of $p_2 = 0.4$, and thus that the optimal decision is d_3. Consider again the additional information $k = 0.9$ for the occurrence of the event "$\Theta = \theta_1$".

Figure 2 shows which values of μ and σ^2 will result in decisions d_1, d_2, and d_3 respectively.

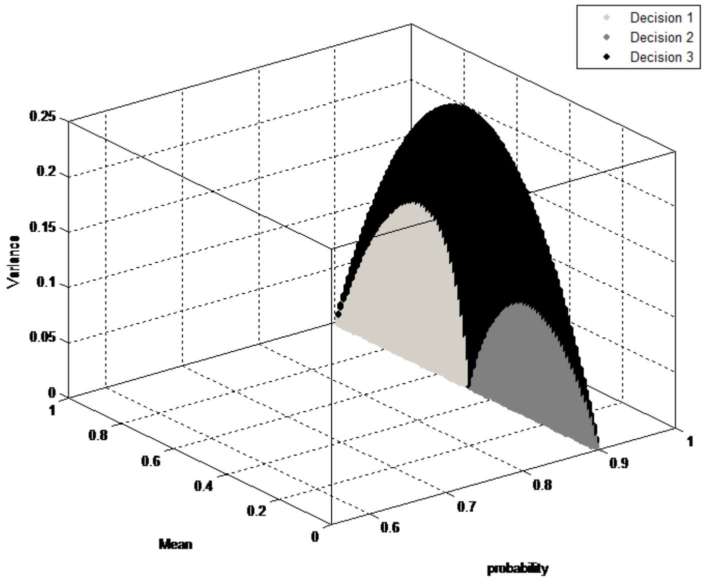

Figure 2: Values of μ and σ^2 that will result in decisions d_1, d_2, and d_3 respectively.

Thus for additional information $k = 0.9$, the values of μ and σ^2 where consensus will be reached between DM_1 and DM_2 are shown in Figure 3.

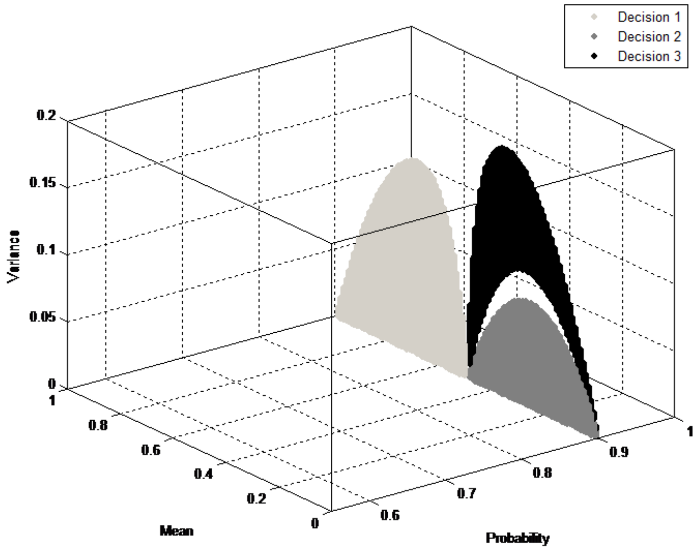

Figure 3: Values of μ and σ^2 where consensus will be reached between DM_1 and DM_2.

Figure 4 shows the values of μ and σ^2 which will lead to consensus between DM_1 and DM_2 for $k = 0.6$, 0.7, 0.8, and 0.9.

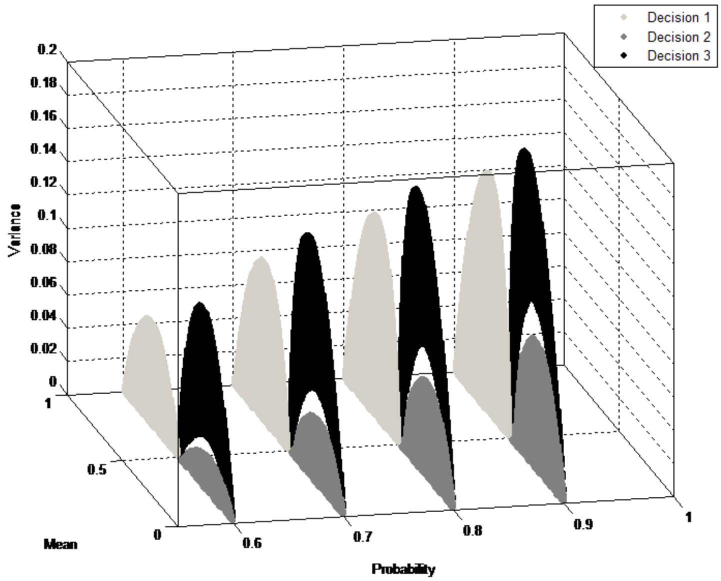

Figure 4: Values of μ and σ^2 where consensus will be reached between DM_1 and DM_2 for $k = 0.6$, 0.7, 0.8, and 0.9.

Considering all values of k, where $1/2 \leq k \leq 1$, the similarity between DM_1 and DM_2 for this example is $s_{12} = \frac{|D_{12}|}{|D|} = 0.4507$ and the distance therefore 0.5493.

Example 2. Consider four DMs with prior probabilities 0.7,0.4,0.3, and 0.8 respectively. Suppose these DMs have the following utility functions given in Tables 5 to 8:

Table 5: Utility function for DM_1.

DM_1	d_1	d_2	d_3
θ_1	5	2	4
θ_2	1	4	3

Table 6: Utility function for DM_2.

DM_2	d_1	d_2	d_3
θ_1	11	6	10
θ_2	3	5	4

Table 7: Utility function for DM_3.

DM_3	d_1	d_2	d_3
θ_1	10	2	6
θ_2	3	9	4

Table 8: Utility function for DM_4.

DM_4	d_1	d_2	d_3
θ_1	4	1	3
θ_2	0	4	3

Using the prior probabilities, the optimal decisions for the DMs are d_1, d_3, d_2, and d_1 respectively.

The similarities between the DMs can be calculated.

Table 9: Similarities between the DMs.

	Similarity
DM_1 and DM_2	$s_{12} = 0.4507$
DM_1 and DM_3	$s_{13} = 0.2666$
DM_1 and DM_4	$s_{14} = 0.9285$
DM_2 and DM_3	$s_{23} = 0.2981$
DM_2 and DM_4	$s_{24} = 0.3792$
DM_3 and DM_4	$s_{34} = 0.2485$

The weight assigned to each DM can also be found (Table 10).

Table 10: Weight assigned to each DM.

	Weight
DM_1	$w_1 = 0.3200$
DM_2	$w_2 = 0.2193$
DM_3	$w_3 = 0.1581$
DM_4	$w_4 = 0.3026$

The highest weight is assigned to DM_1, so DM_1 is most in agreement with the rest of the group members.

4. The general case of more than two outcomes

Consider a parameter and decision space denoted by $\Theta = \{\theta_1, \theta_2, \cdots, \theta_m\}$ and $D = \{d_1, d_2, \cdots, d_n\}$ respectively. DM_i's utilities are given in Table 11.

Table 11: General utility function for DM_i.

		Decision space				
		d_1	d_2	$\cdots$	d_n	d_E
Parameter space	θ_1	U_{i11}	U_{i12}	$\cdots$	U_{i1n}	U_{i1E}
	θ_2	U_{i21}	U_{i22}	$\cdots$	U_{i2n}	U_{i2E}
	$\vdots$					
	θ_m	U_{im1}	U_{im2}	$\cdots$	U_{imn}	U_{imE}

As in Section 3 the decision d_E has been added to the decision space for DM_i where

$$U_{irE} = \sum_{j=1}^{n} P\left[\text{choose } d_j \,|\Theta = \theta_r\right] U_{irj} \;\; i = 1, \ldots, l, \; r = 1, \ldots, m.$$

The information provided by an expert is now in the form of probabilities $k_1, k_2, \cdots, k_{m'}$, where $m' = m-1$, $k_r = P\left(\text{“}\Theta = \theta_r\text{”}\right)$, and $\sum_{r=1}^{m} k_r = 1$. According to Zapata-Vázquez et al. (2014) eliciting expert knowledge about several quantities is a complex task when the quantities exhibit associations as in the above.

There is a growing use of elicitation to express expert knowledge about uncertain quantities in the form of probability distributions. For this general case the assumption is made that the expert's knowledge can be represented by a Dirichlet distribution conditional on θ_r. Thus the distribution of $\boldsymbol{k} = (k_1, k_2, \cdots, k_{m'})$ given $\Theta = \theta_r$, $r = 1, 2, \cdots, m$ is a Dirichlet distribution with parameter vector $\boldsymbol{\alpha}_r = (\alpha_{r1}, \alpha_{r2}, \cdots, \alpha_{rm})$ and probability density function

$$f\left(\boldsymbol{k}|\Theta = \theta_r\right) = c\left(\boldsymbol{\alpha}_r\right) \prod_{j=1}^{m} k_j^{\alpha_{rj}-1}$$

for $k_j > 0$, $\sum_{j=1}^{m} k_j = 1$, and $c\left(\boldsymbol{\alpha}_r\right) = \dfrac{\Gamma\left(\sum_{i=1}^{m} \alpha_{ri}\right)}{\prod_{i=1}^{m} \Gamma\left(\alpha_{ri}\right)}$, $r = 1, 2, \cdots, m$.

The means and variances of $k_1, k_2, \cdots, k_m$, are given, respectively, by

$$E\left(k_j \left|\alpha_{rj}\right.\right) = \frac{\alpha_{rj}}{N_r}$$

and

$$\operatorname{Var}\left(k_j \left|\alpha_{rj}\right.\right) = \frac{\alpha_{rj}\left(N_r - \alpha_{rj}\right)}{N_r^2\left(N_r + 1\right)},$$

where $N_r = \sum_{i=1}^{m} \alpha_{ri}$. The k_js are correlated and

$$\mathrm{Cov}\left(k_i, k_j \left| \alpha_{ri}, \alpha_{rj}\right.\right) = -\frac{\alpha_{ri}\alpha_{rj}}{N_r^2\left(N_r+1\right)}.$$

Consider the first $s < m$ elements of $\boldsymbol{k}$ and let $k_{s+1}^{\star} = \sum_{i=s+1}^{m} k_i = 1 - \sum_{i=1}^{s} k_i$. Then according to the marginal property the distribution of $\left(k_1, k_2, \cdots, k_s, k_{s+1}^{\star}\right)$ is Dirichlet, conditional on θ_r, with parameter vector $\left(\alpha_{r1}, \alpha_{r2}, \cdots, \alpha_{rs}, \alpha_{rs+1}^{\star}\right)$ where $\alpha_{rs+1}^{\star} = \sum_{i=s+1}^{m} \alpha_{ri} = N_r - \sum_{i=1}^{s} \alpha_{ji}$. The marginal distribution of k_i is then the beta distribution with parameters α_{ri} and $N_r - \alpha_{ri}$.

The Dirichlet family is sometimes parameterised with $p_{ri} = \frac{\alpha_{ri}}{N_r}$, $i = 1, 2, \cdots, m$ and N_r. According to this parameterisation the p_{ri}s control the means of the k_is, while N_r control the overall amount of uncertainty. We could therefore ask the expert to provide a judgement of the expected value of each k_i conditional on θ_r together with one further judgement concerned with uncertainty to identify N_r.

According to Zapata-Vázquez et al. (2014) methods that have been proposed in the literature for eliciting a Dirichlet distribution can mostly be viewed as suggesting alternative kinds of judgement of location to identify the p_{ri}s and alternative kinds of judgement of uncertainty to identify N_r. For instance, the expert could be asked for a median value of k_i because the judgement of equal probability is generally made more accurately.

One way to identify N_r is simply by eliciting a measure of uncertainty about a single k_i. Other approaches elicit N_r as a measure of the amount of information that the expert have. In Garisch and Groenewald (2007) the past performance of the expert is used to elicit a beta distribution. This can be extended to the elicitation of a Dirichlet distribution.

If an expert is consulted, each *DM*'s prior probability $p_j = P(\Theta = \theta_j)$ can be updated and the posterior probability p'_j calculated where $\boldsymbol{k}\,|\Theta = \theta_r \sim \mathrm{Dirichlet}(\alpha_{r1}, \alpha_{r2}, \cdots, \alpha_{rm})$.

Then

$$\begin{aligned} p'_j &= P(\Theta = \theta_j \,|\boldsymbol{k}) \\ &= \frac{f\left(\boldsymbol{k}\left|\Theta = \theta_j\right.\right) p_j}{\sum_{t=1}^{m} f\left(\boldsymbol{k}\left|\Theta = \theta_t\right.\right) p_t} \\ &= \frac{c\left(\boldsymbol{\alpha}_j\right) \prod_{i=1}^{m} k_i^{\alpha_{ji}-1} p_j}{\sum_{t=1}^{m} \left(c\left(\boldsymbol{\alpha}_t\right)\left(\prod_{v=1}^{m} k_v^{\alpha_{tv}-1}\right) p_t\right)}. \end{aligned}$$

By continuing in the same fashion as in Section 3, $(\boldsymbol{k}, \boldsymbol{\mu}, \boldsymbol{\sigma})$ can be calculated such that consensus is reached.

Example 3. Consider the two *DMs* in Example 1 but now with a parameter space with three points, $\Theta = \{\theta_1, \theta_2, \theta_3\}$. Suppose the utility function of DM_1 is the following (Table 12).

Table 12: Utility function for DM_1.

	DM_1	d_1	d_2	d_3
Parameter space	θ_1	5	2	4
	θ_2	1	4	3
	θ_3	2	1	5

(Decisions: d_1, d_2, d_3)

For prior probabilities of 0.7,0.2, and 0.1 that Θ equals θ_1, θ_2, and θ_3 respectively, the optimal decision is d_1.

Suppose DM_2 has the following utility function (Table 13).

Table 13: Utility function for DM_2.

	DM_2	d_1	d_2	d_3
Parameter space	θ_1	11	6	10
	θ_2	3	5	4
	θ_3	2	4	7

(Decisions: d_1, d_2, d_3)

Consider prior probabilities of 0.4,0.1, and 0.5 for DM_2 and thus an optimal decision of d_3.

Suppose additional information $k_1 = 0.9$, $k_2 = 0.05$, and $k_3 = 0.05$ are provided where $k_i = P\left(\text{“}\Theta = \theta_i\text{”}\right)$.

One way to display the results is to consider the posterior probabilities. To calculate the posterior probabilities, the values $(\boldsymbol{k}, \boldsymbol{\mu}, \boldsymbol{\sigma})$ that will result in consensus must first be found. Given the additional information, the posterior probabilities of the DMs for θ_1, θ_2, and θ_3 that will result in consensus are shown in Figure 5.

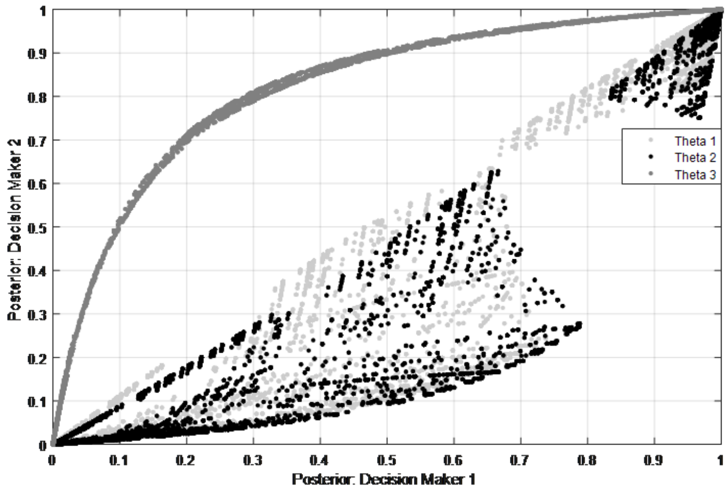

Figure 5: Posterior probabilities of DM_1 and DM_2 resulting in consensus for $\boldsymbol{k} = [0.9, 0.05, 0.05]$.

The posterior probabilities of the *DMs* for θ_1, θ_2, and θ_3 that will result in consensus by choosing decisions 1, 2 and 3 are shown in Figures 6, 7, and 8 respectively.

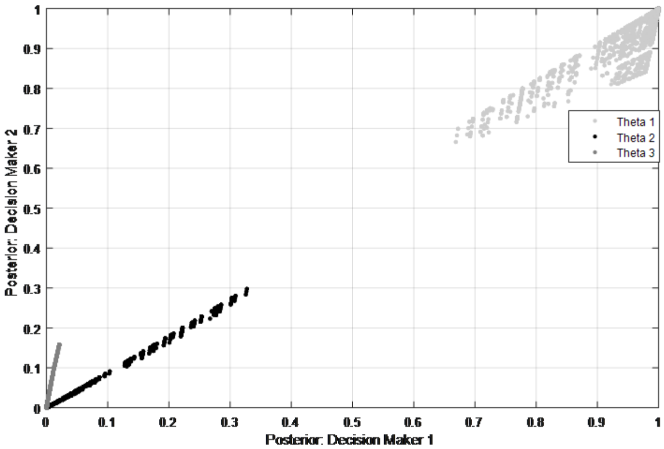

Figure 6: Consensus, choosing decision 1, for $\boldsymbol{k} = [0.9, 0.05, 0.05]$.

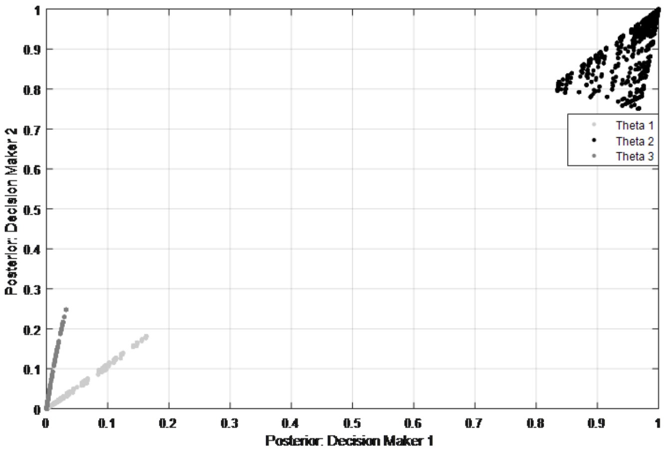

Figure 7: Consensus, choosing decision 2, for $\boldsymbol{k} = [0.9, 0.05, 0.05]$.

The similarity between DM_1 and DM_2 can be calculated. This can be repeated for all *DM*'s and used in the assignment of weights. For this example the similarity between DM_1 and DM_2 is 0.7563.

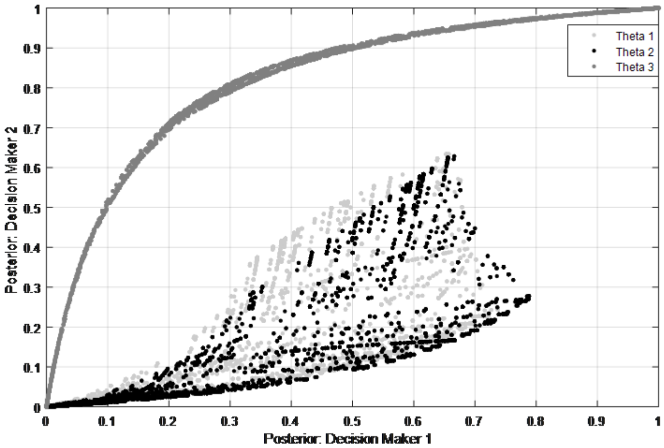

Figure 8: Consensus, choosing decision 3, for $\boldsymbol{k} = [0.9, 0.05, 0.05]$.

5. A Compromise decision

According to "Making Group Decisions" (eXtension.org, 2014) a compromise or randomised decision is applicable when there are two or more distinct options and the *DMs* are not in agreement. A middle position is then created that incorporates ideas from all sides.

According to Bossert and Tan (1995) game theory has provided two approaches to compromise decisions, namely, axiomatic and strategic models. In the axiomatic approach, originating in Nash (1950), solution concepts are derived from properties that are considered desirable. In the strategic approach, bargaining problems are formulated explicitly as extensive form games (Rubenstein, 1982).

The solution by compromise, presented by Nash (1950), proceeds logically from certain weak assumptions to obtain a surprisingly strong conclusion. According to Bossert and Tan (1995), Nash's game consists of a single stage in which the players simultaneously announce "demands" in terms of utilities. If these demands are compatible given the set of feasible utility vectors, then each player receives the amount he or she demanded; otherwise the disagreement event occurs.

In the compromise or randomised solution presented by Nash (1950) a $\boldsymbol{\delta}$ must be chosen such that $P(\boldsymbol{\delta}) = \prod_{i=1}^{l} \left[\boldsymbol{\delta}'\boldsymbol{U}_i\right]^{\gamma_i}$ is a maximum, where $\boldsymbol{\delta}$ is the randomised decision, $\boldsymbol{U}_i$ is the vector of expected utilities for DM_i, and equal weights $\gamma_i = 1/l$ are assigned to the *DMs*.

Kalai (1977) proposes a non-symmetric Nash solution. This solution satisfies all of Nash's axioms (Weerhandi and Zidek, 1981) except the axiom of symmetry. Thus, a weight $\gamma_i \geq 0$ is assigned to DM_i where $\sum_{i=1}^{l} \gamma_i = 1$.

One method to determine the weights γ_i would be to call upon the *DMs* to agree in preliminary discussions on a choice of γ_i. This would permit an individual i who lacked confidence in his own judgment to defer to the group by accepting a small value for $\gamma_i < 1/l$. The method proposed in this paper is to use the measure of similarity defined in Section 3.

Example 4. Consider again the four *DMs* in Example 2. The expected utilities and assigned weights are given in Table 14

Table 14: Expected utilities and assigned weights.

	Expected Utilities			Weights	
	d_1	d_2	d_3	Symmetric	Non-symmetric
DM_1	3.8	2.6	3.7	0.25	0.3200
DM_2	6.2	5.4	6.4	0.25	0.2193
DM_3	5.1	6.9	4.6	0.25	0.1581
DM_4	3.2	1.6	3.0	0.25	0.3026

The symmetrical Nash solution is displayed in Figure 9.

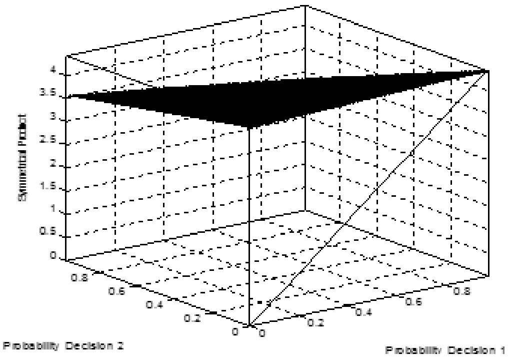

Figure 9: The symmetrical Nash solution.

The maximum value is 4.4282 and the symmetrical randomised decision is to choose d_1 with probability 1.

Figure 10 shows the non-symmetrical solution.

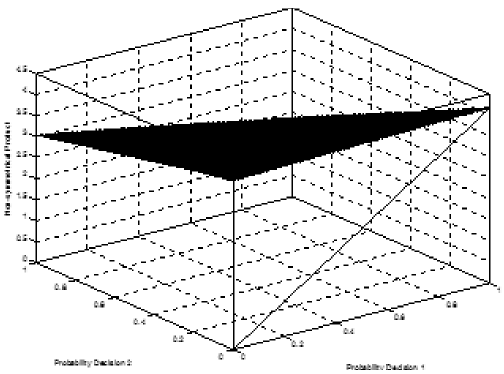

Figure 10: The non-symmetrical Nash solution.

The maximum value is 4.1663 and the non-symmetrical randomised decision is the same as the symmetrical one.

Thus although there is a difference in the maximum values of the symmetrical and non-symmetrical cases, the randomised decisions are the same and d_1 is chosen with probability 1.

Table 15 shows the utilities that can be expected by each *DM* for the original and the symmetrical/non-symmetrical cases.

Table 15: Expected utilities.

	Decision Makers			
	DM_1	DM_2	DM_3	DM_4
Original	3.8	6.4	6.9	3.2
Symmetrical/Non-symmetrical	3.8	6.2	5.1	3.2

The first row in Table 15 represents the expected utilities of the *DMs* if they were to make individual decisions. The table shows that by negotiating and finding a consensus solution through compromise, there are for two *DMs* a decline in the expected utilities. It is always the case that the expected utilities for the compromise decision will be less or equal to those of the individual ones. By finding a solution through compromise, the outcome will be something that not all will be totally satisfied with, but that everyone can live with.

6. Conclusion

In this paper l decision makers, each with his/her own utility function, are considered. A mutually acceptable decision must be found from n possible ones. The prior probabilities of the *DMs* can be updated using the information provided by an expert which is presented in the form of a probability. Experts with probabilities for consensus greater than some cut-off value are then consulted. The quality of the information received from the expert is measured using the beta distribution for the case of two possible outcomes and the more general Dirichlet distribution for more than two. The Jaccard Similarity Coefficient is used to measure similarity or diversity and used to assign weights to the *DMs*. These weights indicate the importance of the *DMs* in the group. The method suggested in this paper to calculate the weights should be viewed as one of a number of approaches that can be considered and not as optimal in general. However, this method can produce an optimal result for a specific case depending on how optimality is defined and on the experts available. Once the weights are calculated, the compromise solution presented by Nash (1950) is considered.

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