A NOTE ON BROWNIAN AREAS AND ARCSINE LAWS

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Abstract: Firstly, we provide simple elementary proofs to derive the exact distributions of the areas under functions of a Brownian motion process and a Brownian bridge process. In the latter case, a solution is therefore provided to a question raised recently in the Mathematics community on StackExchange (`http://math.stackexchange.com/questions/1006101`). These random areas often occur in statistical applications and play an important role in, for example, financial mathematics. Comparisons are made between the variances of the two random areas, deriving interesting results that appear to be new in the statistical literature. Some illustrative examples are provided. Secondly, we derive a new arcsine law for a standard Brownian bridge process.

1. Introduction

Throughout the discussion below we restrict ourselves to versions of a standard Brownian motion process $\{B(t), 0 \le t \le T\}$ and a standard Brownian bridge process $\{B^0(t), 0 \le t \le T\}$ defined on a finite interval $[0,T]$. The latter process can be constructed by letting

$$B^0(t) := B(t) - \frac{t}{T}B(T), \qquad 0 \le t \le T. \tag{1}$$

Note that $B^0(0) = B^0(T) = 0$, and

$$\operatorname{Cov}\left(B^0(s), B^0(t)\right) = \min(s,t) - \frac{st}{T},$$

for all $0 \le s,t \le T$.

It is well known that, by applying Itô's lemma, the following two stochastic integrals can be expressed as

$$\int_0^t B(s)dB(s) = \frac{B^2(t)}{2} - \frac{t}{2}, \qquad 0 \le t \le T,$$
$$\int_0^t B^2(s)dB(s) = \frac{B^3(t)}{3} - \int_0^t B(s)\, ds, \qquad 0 \le t \le T.$$

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It is often remarked in the statistical literature that the random Riemann integral $A(t) := \int_0^t B(s)ds$ cannot be expressed in a more explicit form and in order to obtain an impression of the behaviour of the sample paths of $A(t)$, for $0 \le t \le T$, one has to rely on Monte-Carlo simulations. This is usually accomplished by simulating Brownian paths in a straightforward manner using the time discretization methodology based on the independent increment property of Brownian motion. The time interval $[0,T]$ may be divided into s subintervals, and then $B(t_{k+1})$ at a specific time $t_{k+1} = (k+1)T/s$, $k = 0,1,\ldots,s-1$, is given by

$$B(t_{k+1}) = \sum_{i=0}^{k} \sqrt{\frac{T}{s}} Z_i,$$

where $Z_0, Z_1, \ldots$, are i.i.d. $N(0,1)$ random variables.

An alternative efficient approach is to apply the Karhunen-Loève expansion of a Brownian motion process. This technique is, for example, frequently used in mathematical finance for pricing a path-dependent financial derivative, such as a continuously monitored Asian option (see Niu and Hickernell (2010) and the references therein). By solving the eigenvalue problem of the covariance operator of $B(t)$, i.e., $\mathrm{Cov}(B(s), B(t)) = \min(s,t)$, the Brownian motion $B(t)$ can be expanded (e.g., see Breiman (1968, p. 261)) as an infinite, uniformly convergent series on $[0,T]$:

$$B(t) = \frac{t}{\sqrt{T}} Z_0 + \frac{\sqrt{2T}}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi t/T)}{n} Z_n. \tag{2}$$

Similarly, from (1) and (2) we derive a Karhunen-Loève expansion for a standard Brownian bridge process:

$$B^0(t) = \frac{\sqrt{2T}}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi t/T)}{n} Z_n. \tag{3}$$

Since the infinite series in (2) and (3) converges uniformly on $[0,T]$ and the individual terms of the series are almost everywhere continuous, the series may be integrated term by term, yielding closed-form expressions for the sample paths of $A(t)$ and $A^0(t) := \int_0^t B^0(s)ds$,

$$A(t) = \frac{t^2}{2\sqrt{T}} Z_0 + \frac{T\sqrt{2T}}{\pi^2} \sum_{n=1}^{\infty} \frac{(1-\cos(n\pi t/T))}{n^2} Z_n, \tag{4}$$

and

$$A^0(t) = \frac{T\sqrt{2T}}{\pi^2} \sum_{n=1}^{\infty} \frac{(1-\cos(n\pi t/T))}{n^2} Z_n. \tag{5}$$

Figure 1 presents graphs of the sample paths of $A(t)$ and $A^0(t)$, calculated from (4) and (5) for $T = 1$ and $T = 2$. The difference between the variability of the sample paths of $A(t)$ and $A^0(t)$ is quite evident. This phenomenon is further explored below for more general functionals of $B(t)$ and $B^0(t)$.

The paper is organized as follows. In Section 2 the exact distributions of certain Brownian motion and Brownian bridge areas are derived, *thus providing a solution to the question raised in the abstract*. Section 3 provides some interesting variance comparisons together with some illustrative examples. In Section 4 a new arcsine law for a standard Brownian bridge process is proved. Section 5 contains alternative proofs of our main theorems.

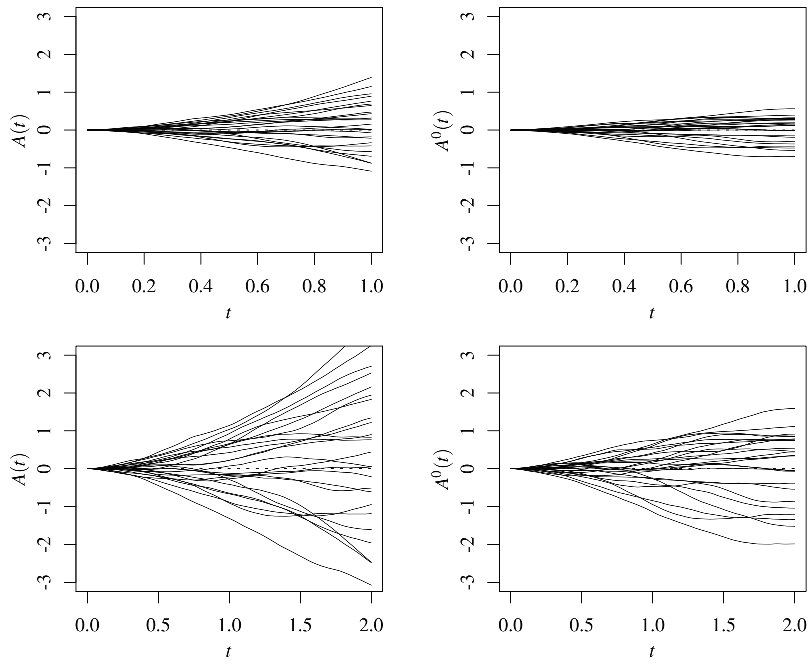

Figure 1: Graphs of $A(t)$ and $A^0(t)$ for $T = 1$ (first row) and $T = 2$ (second row).

2. Distribution of Brownian Motion and Brownian Bridge Areas

In this section we consider the following two random Riemann integrals,

$$A(t) := \int_0^t h(s)B(s)ds, \qquad 0 \le t \le T,$$

and

$$A^0(t) := \int_0^t h(s)B^0(s)ds, \qquad 0 \le t \le T,$$

for some deterministic real-valued function $h : [0,T] \to \mathbb{R}$.

In the next lemma $A(t)$ and $A^0(t)$ are expressed in terms of Itô stochastic integrals, *which are results of independent interest*, and will be applied to derive the exact distributions of $A(t)$ and $A^0(t)$ in Theorem 1 below.

Lemma 1 Let $h : [0,T] \to \mathbb{R}$ be a continuous function in $L^1([0,T])$. It then follows that

$$A(t) = \int_0^t \left(\int_s^t h(x)dx \right) dB(s), \qquad 0 \le t \le T, \tag{6}$$

and

$$A^0(t) = \int_0^t \left(\int_s^t h(x)dx \right) dB^0(s), \qquad 0 \le t \le T. \tag{7}$$

Proof. Applying Itô's lemma, the following integration-by-parts formula holds for $0 \le t \le T$,

$$\int_0^t \left(\int_0^s h(x)dx \right) dB(s) = \left(\int_0^t h(x)dx \right) B(t) - \int_0^t h(s)B(s)ds.$$

This equation can be rewritten as

$$\begin{aligned} A(t) := \int_0^t h(s)B(s)ds &= \left(\int_0^t h(x)dx \right) \int_0^t dB(s) - \int_0^t \left(\int_0^s h(x)dx \right) dB(s) \\ &= \int_0^t \left(\int_0^t h(x)dx - \int_0^s h(x)dx \right) dB(s) \\ &= \int_0^t \left(\int_s^t h(x)dx \right) dB(s), \end{aligned}$$

which proves (6). Using this identity we also have that

$$\begin{aligned} A^0(t) &= A(t) - \frac{B(T)}{T} \int_0^t h(s)sds \\ &= \int_0^t \left(\int_s^t h(x)dx \right) dB(s) - \frac{B(T)}{T} \int_0^t \int_s^t h(x)dxds \\ &= \int_0^t \left(\int_s^t h(x)dx \right) d\left(B(s) - \frac{B(T)}{T}s \right) \\ &= \int_0^t \left(\int_s^t h(x)dx \right) dB^0(s), \end{aligned}$$

which proves (7). ■

Theorem 1 Let $h : [0,T] \to \mathbb{R}$ be a continuous function in $L^2([0,T])$. Then, for $0 \le t \le T$, $A(t)$ and $A^0(t)$ are normally distributed with zero expectations and variances $\sigma^2(t)$ and $\sigma_0^2(t)$ respectively, where

$$\sigma^2(t) = \int_0^t \left(\int_s^t h(x)dx \right)^2 ds, \tag{8}$$

and

$$\sigma_0^2(t) = \sigma^2(t) - \frac{1}{T} \left(\int_0^t xh(x)dx \right)^2. \tag{9}$$

Proof. It immediately follows from (6) and Itô's stochastic integral calculus that $A(t)$ is normally distributed with expectation zero and variance $\sigma^2(t)$, by the isometry property.

Clearly, from (1), (6) and (7) we conclude that $A^0(t)$ is Gaussian with expectation zero. As far as the variance of $A^0(t)$ is concerned, note that from the definition of $B^0(t)$ it follows that for $0 \le t \le T$,

$$A(t) = A^0(t) + \frac{B(T)}{T} \int_0^t xh(x)dx. \tag{10}$$

Furthermore, $\{B^0(s), 0 \le s \le t\}$ is independent of $B(T)$, since $(B^0(s), B(T))$ is a Gaussian random vector with covariance function

$$
\begin{aligned}
\operatorname{Cov}\left(B^0(s), B(T)\right) &= \mathrm{E}\left(B^0(s)B(T)\right) \\
&= \mathrm{E}\left(\left(B(s) - \frac{s}{T}B(T)\right)B(T)\right) \\
&= \min(s,T) - \frac{s}{T}\,\mathrm{E}\left(B^2(T)\right) \\
&= s - \frac{s}{T}T = 0.
\end{aligned}
$$

This implies that $A^0(t)$ is independent of $B(T)$, $0 \le t \le T$. Taking variances on both sides of (10) immediately yields the result stated in (9). ■

Remarks

- For the special case $h(x) \equiv 1$, we conclude from the theorem that

$$A(t) \stackrel{d}{=} N\left(0, \frac{t^3}{3}\right), \qquad 0 \le t \le T,$$

  a result known in the literature, usually derived by proving the convergence of certain Riemann sums.

- Also, if $h(x) \equiv 1$,

$$A^0(t) \stackrel{d}{=} N\left(0, \frac{t^3}{3}\left(1 - \frac{3t}{4T}\right)\right), \qquad 0 \le t \le T,$$

  a result that seems to be new, thus providing a solution to the question raised in the Mathematics community on StackExchange (`http://math.stackexchange.com/questions/1006101`).

- Further, note the interesting relation that

$$\frac{\sigma_0^2(1)}{\sigma^2(1)} = \frac{1}{4}, \tag{11}$$

  if $h(x) \equiv 1$.

3. Variance Comparisons

Consider the ratio

$$
\begin{aligned}
r(t) &:= \frac{\sigma_0^2(t)}{\sigma^2(t)}, \qquad 0 \le t \le T, \\
&= 1 - \frac{\left(\int_0^t \int_s^t h(x)dxds\right)^2 / T}{\sigma^2(t)}, \\
&=: 1 - g(t).
\end{aligned}
$$

Applying L'Hôpital's rule, it immediately follows that $g(0)=0$. Also, it readily follows that $g'(t)\geq 0$ if and only if $t/T \geq g(t)$ for all $0\leq t\leq T$. This implies, since $g(t)$ is continuous on $[0,T]$, that

$$T = \arg \sup_{0\leq t\leq T} g(t),$$

so that

$$\inf_{0\leq t\leq T} r(t) = 1-g(T) = r(T).$$

We now present two illustrative examples.

Example 1. Choose $h(x)=x^r$, $r\geq 0$. After some algebra we obtain the interesting result that for *all* $T>0$,

$$r(T) = \frac{1}{2(r+2)},$$

which agrees with the result stated in (11) if $r=0$.

Example 2. Consider $h(x)=\cos x$. After some tedious calculations we obtain

$$r(T) = \frac{2T-\sin(2T)-4(1-\cos T)^2/T}{2T+3\sin(2T)+4T(1-\cos^2 T)-8\sin T}.$$

The function $r(T)$ has curious, but interesting behaviour as can be seen from the following values:

$$r\left(\frac{\pi}{2}\right) = \frac{\pi^2-8}{\pi(3\pi-8)} \cong 0.418,$$
$$r(\pi) = \frac{\pi^2-8}{\pi^2} \cong 0.189,$$
$$r\left(\frac{3\pi}{2}\right) = \frac{9\pi^2-8}{3\pi(9\pi+8)} \cong 0.236,$$

and, surprisingly, $r(2\pi k)=1$ for $k=1,2,\ldots$, which means that the variances of $A(T)$ and $A^0(T)$ are equal when T is a multiple of 2π.

4. An Arcsine Law for a Standard Brownian Bridge Process

In the theory of Stochastic Processes, the *arcsine laws* are a collection of results for one-dimensional random walks and Brownian motion. The best known of these is attributed to Lévy (1939).

All three laws relate path properties of a Brownian motion process to the arcsine distribution. A random variable X on $[0,1]$ is arcsine-distributed if

$$\mathrm{P}(X\leq x) = \frac{2}{\pi}\arcsin\sqrt{x}.$$

The first arcsine law states that the *proportion* of time that the one-dimensional Brownian motion process $\{B(t), 0\leq t\leq 1\}$ is positive follows an arcsine distribution. The second arcsine law describes the distribution of the *last* time $\{B(t), 0\leq t\leq 1\}$ changes sign. The third arcsine law states that the time at which $\{B(t), 0\leq t\leq 1\}$ achieves its *maximum* is arcsine distributed. It is well known that the second and third laws are equivalent.

Similar arcsine laws for a Brownian bridge process $\{B^0(t), 0 \leq t \leq T\}$ apparently do not exist in the literature, although they can easily be derived. For example, in Theorem 2 below we prove an arcsine law which is analogous to the second arcsine law for a Brownian motion process.

In order to prove the theorem we require the following lemma.

Lemma 2 Let X be a standard $N(0,1)$ random variable with distribution function Φ, then for any constant $c \geq 0$ we have

$$\mathrm{E}\{\Phi(cX)I(X \geq 0)\} = \frac{1}{4} + \frac{1}{2\pi}\arccos\frac{1}{\sqrt{1+c^2}},$$

where $I(\cdot)$ denotes the indicator function.

Proof. Let ϕ be the standard normal density function, then

$$\begin{aligned}
\mathrm{E}\{\Phi(cX)I(X \geq 0)\} &= \int_0^\infty \Phi(cx)\phi(x)dx \\
&= \int_0^\infty \int_{-\infty}^{cx} \phi(y)\phi(x)dydx \\
&= \int_0^\infty \int_{-\infty}^{0} \phi(y)\phi(x)dydx + \int_0^\infty \int_0^{cx} \phi(y)\phi(x)dydx \\
&= \frac{1}{4} + \frac{1}{2\pi}\int_0^\infty \int_0^{cx} e^{-\frac{1}{2}(y^2+x^2)}dydx.
\end{aligned}$$

Using polar co-ordinates by setting $x = r\cos\theta$ and $y = r\sin\theta$ and noticing that the Jacobian of the transformation is given by

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r,$$

we deduce that

$$\begin{aligned}
\mathrm{E}\{\Phi(cX)I(X \geq 0)\} &= \frac{1}{4} + \frac{1}{2\pi}\int_0^{\arctan c}\int_0^\infty re^{-\frac{1}{2}r^2}drd\theta \\
&= \frac{1}{4} + \frac{1}{2\pi}\arctan c \\
&= \frac{1}{4} + \frac{1}{2\pi}\arccos\frac{1}{\sqrt{1+c^2}}.
\end{aligned}$$

■

In the proof of the theorem below we will apply the following known facts:

(i) If $\{B(t), t \geq 0\}$ is a standard Brownian motion process then $\{-B(t), t \geq 0\}$ and $\{B(t+c) - B(c), t \geq 0\}$ are also standard Brownian motion processes, for some finite constant $c \geq 0$.

(ii) A standard Brownian bridge process on $[0,T]$ can also be represented as

$$B^0(t) = \left(\frac{T-t}{\sqrt{T}}\right)B\left(\frac{t}{T-t}\right), \qquad 0 \leq t \leq T.$$

(iii) The density function of $B^0(t)$ is $N(0,\sigma_T^2(t))$, where

$$\sigma_T^2(t) := \frac{t(T-t)}{T}, \qquad 0 \le t \le T.$$

(iv) $\mathrm{P}\left(\sup_{0\le s\le t} B(s) \ge x\right) = 2\,\mathrm{P}\left(B(t) \ge x\right),\ x \in \mathbb{R}.$

We now derive an arcsine law that describes the distribution of the last time a standard Brownian bridge process $B^0(\cdot)$ changes sign in the time interval $[0,a]$, $0 \le a \le T$.

Define the stopping time

$$\tau := \sup\left\{0 \le t \le a : B^0(t) = 0\right\}.$$

Theorem 2 For $0 \le x < a \le T$ we have that

$$\mathrm{P}\left(\tau \le x\right) = \frac{2}{\pi} \arcsin \sqrt{\frac{x(T-a)}{a(T-x)}}.$$

Proof. For ease of notation, write $\alpha(y) := y/(T-y)$, $0 \le y \le T$. Then

$$\begin{aligned}
\mathrm{P}\left(\tau \le x\right) &= \mathrm{P}\left(B^0(\cdot) \text{ has no zeros in } (x,a]\right) \\
&= 1 - \mathrm{P}\left(B^0(\cdot) \text{ has at least one zero in } (x,a]\right) \\
&= 1 - \int_{-\infty}^{\infty} \mathrm{P}\left(B^0(\cdot) \text{ has at least one zero in } (x,a] | B^0(x) = y\right) f_{B^0(x)}(y)dy \\
&= 1 - \int_{-\infty}^{0} \mathrm{P}\left(\sup_{x<s\le a} B^0(s) \ge 0 \middle| B^0(x) = y\right) f_{B^0(x)}(y)dy \\
&\quad - \int_{0}^{\infty} \mathrm{P}\left(\inf_{x<s\le a} B^0(s) \le 0 \middle| B^0(x) = y\right) f_{B^0(x)}(y)dy \\
&=: 1 - (\mathscr{I}_1 + \mathscr{I}_2).
\end{aligned}$$

Applying (i)–(iv) and the independence of Brownian motion increments we deduce that

$$\begin{aligned}
\mathscr{I}_1 &= \int_{-\infty}^{0} \mathrm{P}\left(\sup_{x<s\le a} B\left(\frac{s}{T-s}\right) \ge 0 \middle| B^0(x) = y\right) f_{B^0(x)}(y)dy \\
&= \int_{-\infty}^{0} \mathrm{P}\left(\sup_{\alpha(x)<s\le\alpha(a)} B(s) \ge 0 \middle| B(\alpha(x)) = \frac{\sqrt{T}}{(T-x)}y\right) f_{B^0(x)}(y)dy \\
&= \int_{-\infty}^{0} \mathrm{P}\left(\sup_{\alpha(x)<s\le\alpha(a)} \Big(B(s) - B(\alpha(x))\Big) \ge -\frac{\sqrt{T}}{(T-x)}y\right) f_{B^0(x)}(y)dy
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{0} \mathrm{P}\left(\sup_{0<s\le\alpha(a)-\alpha(x)} \Big(B(s+\alpha(x))-B(\alpha(x))\Big) \ge -\frac{\sqrt{T}}{(T-x)}y\right) f_{B^0(x)}(y)dy \\
&= \int_{-\infty}^{0} \mathrm{P}\left(\sup_{0<s\le\alpha(a)-\alpha(x)} B(s) \ge -\frac{\sqrt{T}}{(T-x)}y\right) f_{B^0(x)}(y)dy \\
&= \int_{0}^{\infty} \mathrm{P}\left(\sup_{0<s\le\alpha(a)-\alpha(x)} B(s) \ge \frac{\sqrt{T}}{(T-x)}y\right) f_{B^0(x)}(y)dy \\
&= 2\int_{0}^{\infty} \mathrm{P}\left(B\Big(\alpha(a)-\alpha(x)\Big) \ge \frac{\sqrt{T}}{(T-x)}y\right) f_{B^0(x)}(y)dy \\
&= 2\int_{0}^{\infty} \left\{1-\Phi\left(\frac{y\sqrt{T}}{(T-x)\sqrt{\alpha(a)-\alpha(x)}}\right)\right\} f_{B^0(x)}(y)dy \\
&= 2\int_{0}^{\infty} \left\{1-\Phi\left(y\sqrt{\frac{T-a}{(a-x)(T-x)}}\right)\right\} f_{B^0(x)}(y)dy \\
&= 1-2\int_{0}^{\infty} \Phi\left(y\sqrt{\frac{x(T-a)}{T(a-x)}}\right) \phi(y)dy \\
&= 1-2\mathrm{E}\left\{\Phi\left(X\sqrt{\frac{x(T-a)}{T(a-x)}}\right) I(X\ge 0)\right\},
\end{aligned}
$$

where X is a $N(0,1)$ random variable.

Similarly, we have that $\mathscr{I}_2 = \mathscr{I}_1$. Hence,

$$
\mathscr{I}_1 + \mathscr{I}_2 = 2 - 4\mathrm{E}\left\{\Phi\left(X\sqrt{\frac{x(T-a)}{T(a-x)}}\right) I(X\ge 0)\right\}.
$$

Applying Lemma 2 above we conclude that

$$
\mathrm{P}(\tau \le x) = \frac{2}{\pi}\arccos\sqrt{\frac{T(a-x)}{a(T-x)}} = \frac{2}{\pi}\arcsin\sqrt{\frac{x(T-a)}{a(T-x)}}.
$$

■

Remark

Note that if in Theorem 2 we set $a = 1$ and let $T \to \infty$, then

$$
\mathrm{P}(\tau \le x) \to \frac{2}{\pi}\arcsin\sqrt{x},
$$

which agrees with the second arcsine law for a *standard Brownian motion* on $[0,1]$.

5. Alternative Proofs

The proof of Theorem 1 is essentially based on Itô's lemma, the isometry property of an Itô stochastic integral, and the interesting fact that $\{B^0(s),\ 0\le s\le t\}$ is independent of $B(T)$. The referee pointed out that it could also be proved by the usual approximation of an integral by a Riemann sum. To broaden the reader's perspective on the subject, we provide such a proof.

Proof. *(Alternative proof of Theorem 1)* Let $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ be any equal-spaced partition of the interval $[0,t]$, i.e., $t_k := (k/n)t$, for $k = 0,1,\ldots,n$. Set $\Delta t := t_k - t_{k-1} = t/n$. We then have, with probability one, that

$$A(t) := \lim_{n\to\infty} \Delta t \sum_{k=1}^{n} h(t_k)B(t_k) =: \lim_{n\to\infty} I_n(t).$$

Clearly, $I_n(t) \stackrel{d}{=} N(0,\sigma_n^2(t))$, where

$$\begin{aligned}\sigma_n^2(t) &:= \operatorname{Var}(I_n(t)) \\ &= (\Delta t)^2 \sum_{k=1}^{n}\sum_{\ell=1}^{n} h(t_k)h(t_\ell)\operatorname{Cov}(B(t_k),B(t_\ell)) \\ &= (\Delta t)^2 \sum_{k=1}^{n}\sum_{\ell=1}^{n} h(t_k)h(t_\ell)\min(t_k,t_\ell).\end{aligned}$$

The symbol $\stackrel{d}{=}$ denotes equality in distribution. Hence, as $n \to \infty$, we obtain that

$$\begin{aligned}\sigma_n^2(t) &\to \int_0^t \int_0^t h(x)h(y)\min(x,y)\,dy\,dx \\ &= \int_0^t h(x)\int_0^x yh(y)\,dy\,dx + \int_0^t xh(x)\int_x^t h(y)\,dy\,dx \\ &= 2\int_0^t h(x)\int_0^x yh(y)\,dy\,dx,\end{aligned}$$

by interchanging the order of integration of the second integral. Further, note that by applying partial integration, $\sigma^2(t)$ defined in (8) can also be written as

$$\sigma^2(t) = 2\int_0^t h(x)\int_0^x yh(y)\,dy\,dx.$$

Let $X_n(t) := I_n(t)/\sigma_n(t)$. Then $X_n(t) \stackrel{d}{=} N(0,1)$, and from Slutsky's theorem we conclude that $I_n(t) = \sigma_n(t)X_n(t) \stackrel{d}{\to} N(0,\sigma^2(t))$. Thus, $A(t) \stackrel{d}{=} N(0,\sigma^2(t))$.

A similar argument as above, replacing $B(\cdot)$ by $B^0(\cdot)$ and recalling that $\operatorname{Cov}(B^0(t_k),B^0(t_\ell)) = \min(t_k,t_\ell) - t_k t_\ell/T$, yields that $A^0(t) \stackrel{d}{=} N(0,\sigma_0^2(t))$, where

$$\sigma_0^2(t) = \sigma^2(t) - \frac{1}{T}\left(\int_0^t xh(x)dx\right)^2.$$

■

The proof of Theorem 2 presented above is mainly based on the fact that $B^0(t)$ can be represented as in (ii), i.e.,

$$B^0(t) = \left(\frac{T-t}{\sqrt{T}}\right) B\left(\frac{t}{T-t}\right), \qquad 0 \le t \le T,$$

and on the result derived in Lemma 2. The author is indebted to the referee for providing the following alternative proof.

Proof. *(Alternative proof of Theorem 2)* We compute $\mathrm{P}(\tau \le x)$ as the sum of contributions from Brownian paths from the left and the right with no zeros in the time interval $(x,a]$. Due to the obvious symmetry these contributions are equal. To be more specific, for $y,z<0$, define

$$u_{\text{left}}(y,z,t) := \mathrm{P}\left(\sup_{0<s\le t} B(s) \le -y,\ B(t) \in dz - y\right),$$

then from Borodin and Salminen (1996, p. 131) it follows that

$$u_{\text{left}}(y,z,t) = \frac{1}{\sqrt{2\pi t}}\left(e^{-(z-y)^2/2t} - e^{-(-z-y)^2/2t}\right).$$

Note that

$$\begin{aligned}
&\mathrm{P}\left(B(\cdot) \text{ has no zeros in } (x,a],\ B(a)\in dz\right)\\
&= \int_{-\infty}^{0} \mathrm{P}\left(B(\cdot) \text{ has no zeros in } (x,a],\ B(a)\in dz \middle| B(x)=\xi\right) f_{B(x)}(\xi)\,d\xi\\
&=: I^{-}(x,z,a).
\end{aligned}$$

Since $B(x)$ is independent of $B_1(s) := B(s+x)-B(x)$ and $\{B_1(s), s\ge 0\}$ is a standard Brownian motion process, we conclude that the left contribution $I^{-}(x,z,a)$ can be written, after applying some algebra, as

$$\begin{aligned}
I^{-}(x,z,a) &= \int_{-\infty}^{0} \mathrm{P}\left(\sup_{x<s\le a} B(s)\le 0,\ B(a)\in dz \middle| B(x)=\xi\right) f_{B(x)}(\xi)\,d\xi\\
&= \int_{-\infty}^{0} \mathrm{P}\left(\sup_{0<s\le a-x} B_1(s)\le -\xi,\ B_1(a-x)\in dz-\xi\right) f_{B(x)}(\xi)\,d\xi\\
&= \int_{-\infty}^{0} u_{\text{left}}(\xi,z,a-x) f_{B(x)}(\xi)\,d\xi\\
&= \frac{e^{-z^2/2a}}{\sqrt{2\pi a}}\,\mathrm{erf}\left(-z\sqrt{\frac{x}{2a(a-x)}}\right),
\end{aligned}$$

where

$$\mathrm{erf}(y) = \frac{1}{\sqrt{\pi}}\int_{-y}^{y} e^{-t^2}\,dt.$$

A similar argument shows that the right contribution $I^{+}(x,z,a)$, for $z>0$, is given by $I^{+}(x,z,a) = I^{-}(x,-z,a)$, as to be expected because of the symmetry mentioned above.

Furthermore, let $f_{B(a)|B(T)}(z|0)$ denote the conditional density of $B(a)$ given $B(T)=0$. We then have that

$$\begin{aligned}
\mathrm{P}(\tau\le x) &= \mathrm{P}\left(B(\cdot) \text{ has no zeros in } (x,a] \middle| B(T)=0\right)\\
&= \int_{-\infty}^{\infty} \mathrm{P}\left(B(\cdot) \text{ has no zeros in } (x,a] \middle| B(a)=z, B(T)=0\right) f_{B(a)|B(T)}(z|0)\,dz.
\end{aligned}$$

Making use of the facts that $B(T)-B(a)$ is independent of $\{B(s),\ s\le a\}$ and that $B(a)=z$ and $B(T)=0$ is equivalent to $B(a)=z$ and $B(T)-B(a)=-z$, we find that

$$\begin{aligned}
\mathrm{P}(\tau\le x) &= \int_{-\infty}^{\infty} \mathrm{P}\left(B(\cdot) \text{ has no zeros in } (x,a] \middle| B(a)=z\right) f_{B(a)|B(T)}(z|0)\,dz \\
&= \int_{-\infty}^{\infty} \left\{\mathrm{P}\left(B(\cdot) \text{ has no zeros in } (x,a], B(a)\in dz\right)/f_{B(a)}(z)\right\} f_{B(a)|B(T)}(z|0)\,dz \\
&= \int_{-\infty}^{0} \left\{I^{-}(x,z,a)/f_{B(a)}(z)\right\} f_{B(a)|B(T)}(z|0)\,dz \\
&+ \int_{0}^{\infty} \left\{I^{+}(x,z,a)/f_{B(a)}(z)\right\} f_{B(a)|B(T)}(z|0)\,dz.
\end{aligned}$$

Clearly, $f_{B(a)|B(T)}(z|0)=f_{B(a)}(z)f_{B(T)-B(a)}(-z)/f_{B(T)}(0)$. Thus, after some tedious calculations and applying Lemma 2, we conclude that

$$\begin{aligned}
\mathrm{P}(\tau\le x) &= 2\sqrt{2\pi T}\int_{-\infty}^{0} I^{-}(x,z,a) f_{B(T)-B(a)}(-z)\,dz \\
&= \frac{2}{\pi}\arcsin\sqrt{\frac{x(T-a)}{a(T-x)}}.
\end{aligned}$$

■

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