DISCRETE DISTRIBUTIONS WITH BATHTUB-SHAPED HAZARD RATES

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Abstract: Discrete distributions with bathtub shaped hazard rates have recently become of interest in reliability modelling and analysis. In the present work, we address the problem of obtaining distributions having such hazard rates when the lifetime is discrete. The methods considered here include discretising continuous bathtub models, construction using the score function, construction from decreasing hazard rate distributions and some other methods currently available in the continuous case. We discuss properties and applications of the discretised quadratic hazard model which has a bathtub shaped hazard rate.

1. Introduction

The origin of distributions with bathtub shaped hazard rates, or bathtub distributions in short, can be traced back to the attempts to model data on bird populations (Deevey, 1947) and to bus motor failure data in Davis (1952), where monotone hazard rate distributions failed to provide reasonable fits. Since then there has been a continuous flow of literature on various types of bathtub distributions. Such distributions are characterised by hazard rates that decrease initially, then remain constant and finally increase rapidly. In most of the work on this topic, lifetime is treated as a continuous random variable. For a review of the literature, discussion and references on bathtub models we refer to Rajarshi and Rajarshi (1988), Lai and Xie (2006) and Nair, Sankaran and Balakrishnan

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(2013). Compared to the voluminous literature in the continuous case, only a limited number of investigations have been carried out when the lifetime X is treated as a discrete random variable. In the sequel, the probability mass function and the reliability function of X are denoted respectively by $f(x) = P[X = x]$ and $S(x) = P[X \geq x]$. The reliability function $S(x)$ is related to the distribution function $F(x) = P[X \leq x]$ as

$$S(x) = 1 - F(x-1).$$

Then the hazard rate of X is defined as,

$$h(x) = P[X = x | X \geq x] = \frac{f(x)}{S(x)}, x = 0, 1, 2...$$

We say that $h(x)$ is bathtub (BT) shaped, (or upside down bathtub (UBT) shaped) if there exist positive integers $1 \leq x_{n_1} \leq x_{n_2} < \infty$ such that $x_1 > x_2 > ... > x_{n_1-1} > x_{n_1} = ... = x_{n_2} < x_{n_2+1} < ...(x_1 < x_2 < ... < x_{n_1} = ... = x_{n_2} > x_{n_2+1} > ...)$, and n_1 and n_2 are called the change points. When there is only one change point, $h(x)$ is decreasing (increasing) on $x = 0, 1, 2...x_0 - 1$ and increasing (decreasing) on $x = x_0, x_0 + 1, ...$ The point x_0 will be referred to as the change point of $h(x)$. When the functional form of $h(x)$ is known, $S(x)$ can be determined from,

$$S(x) = \begin{cases} \prod_{t=0}^{x-1} (1 - h(x)) & : x = 1, 2, ... \\ 1 & : x = 0 \end{cases} \tag{1}$$

Lai and Wang (1995) proposed a discrete power distribution with

$$f(x) = \frac{x^{\alpha}}{\sum_{x=0}^{b} x^{\alpha}}, x = 0, 1, 2, ...b; \alpha \in \mathbf{R}$$

for lifetime random variables and it was proved that $h(x)$ is BT for $\alpha < 0$. The rest of the BT models are of recent origin. The discretised version of the inverse Weibull law was considered in Jazi, Lai and Alamatsaz (2010),

$$S(x) = 1 - q^{(x-1)^{\beta}}, x = 1, 2, 3...; 0 < q < 1; \beta > 0$$

which is UBT. A special case when $\beta = 2$ is discussed in Hussain and Ahmad (2014) called inverse Rayleigh, whose hazard rate can also be UBT with change point at $x = 1$ or 2 for $0 < q < 0.75$ and change point $x_0 = 2$ as $q \to 0$. A competing risks model with hazard rate of the form,

$$h(x) = p + (1 - p)r(x),$$

where $r(x)$ is the hazard rate of an exponential Poisson law, was shown to have a BT shape in Jiang (2010). The discretised version of the modified Weibull distribution with reliability function,

$$S_1(x) = q^{x^{\beta}} c^x, x = 0, 1, 2..., 0 < q < 1, \beta > 0, c \geq 1$$

discussed in Noughabi, Roknabadi and Borzadaran (2011) possess a BT hazard rate. Another Weibull related distribution is the discrete additive Weibull with

$$S_2(x) = q_1^{x^{\alpha}} q_2^{x^{\beta}}, x = 0, 1, 2...; \alpha, \beta > 0,$$

where $q_1 = e^{-\lambda_1}$, $q_2 = e^{-\lambda_2}$, $\lambda_1, \lambda_2 > 0$ presented by Bebbington, Lai and Zitikis (2012). They have studied the shape of the hazard rate and found that if $\alpha < 1 < \beta$, $h(x)$ is BT with the minimum achieved at one of the three points $[t_{\alpha,\beta}]$, $1+[t_{\alpha,\beta}]$ and $2+[t_{\alpha,\beta}]$, where $[t_{\alpha,\beta}]$ is the integer part of

$$t_{\alpha,\beta} = \left(\frac{\alpha(1-\alpha)\lambda_1}{\beta(\beta-1)\lambda_2}\right)^{\frac{1}{\beta-\alpha}}.$$

A similar conclusion holds for $\beta < 1 < \alpha$. Later, Noughabi, Roknabadi and Borzadaran (2013) proposed the discrete modified Weibull extension,

$$S_3(x) = q^{\alpha\left(\exp\left\{\left(\frac{x}{\alpha}\right)^{\beta}\right\}-1\right)}, x = 0, 1, 2..., 0 < q < 1, \beta, \alpha > 0,$$

as a BT distribution. Yet another Weibull extension is the reduced modified Weibull family discussed in Almalki and Nadarajah (2014) with reliability function,

$$S_4(x) = q^{\sqrt{x}(1+bc^x)}, x = 0, 1, 2..., 0 < q < 1, b > 0, c \geq 1.$$

The hazard rate of this distribution is increasing if $bc(c-\sqrt{2}) < \sqrt{2}-1$ and has BT shape otherwise. The limited number of prevailing BT distributions reviewed above appears to be insufficient to model a wide variety of data sets. If the stochastic mechanism that generates the data is known, we need a model that is appropriate for it. Further, the observations may sometimes suggest a BT shape through the empirical hazard rate with a known shape that would require a distribution satisfying this particular shape. All these point to the need for developing some methods of arriving at BT distributions, which do not appear to have been considered so far. This motivates the present investigation. There is a vast amount of literature on the methods of such constructions in the continuous case which can be adopted for use for discrete lifetimes. While this is the case in some of the methods we propose, there are some methods for which there is no counterpart in the continuous case. In the following sections, we discuss the methods for generating bathtub models for discrete data. In Section 2, we introduce a method of constructing bathtub models using the score function. This is followed in Section 3, where we consider discretising continuous bathtub distributions. By appropriately modifying decreasing hazard rates, one can obtain BT models. This is discussed in Section 4. Some methods already available in the continuous case are used in the discrete case as well in Section 5. The discretised quadratic hazard model is studied in detail in Section 6. Section 7 provides conclusions of the study. The data sets are given in Appendix A and the program code is given in Appendix B.

2. Properties of the Score Function

The score function $\eta(x) = \dfrac{-g'(x)}{g(x)}$, where $g(x)$ is the probability density function of the continuous lifetime, has been discussed in Glaser (1980). Its discrete version is,

$$\eta(x) = \frac{f(x) - f(x+1)}{f(x)}, \tag{2}$$

which will be used in this section to offer some methods by which BT and UBT distributions can be constructed. Now we provide a simple result using $\eta(x)$ to determine the shape of $h(x)$.

Theorem 1 The random variable X has BT(UBT) hazard rate if and only if $\eta(x) = h(x+1)$ possesses a unique zero $x_0 > 0$ such that $h(x-1) \geq (\leq) h(x)$ in $[0, x_0)$ and $h(x-1) \leq (\geq) h(x)$ in $[x_0, b)$.

Proof.

$$
\begin{aligned}
h(x+1) - h(x) &= \frac{f(x+1)}{S(x+1)} - \frac{f(x)}{S(x)} = \frac{f(x+1)S(x) - f(x)S(x+1)}{S(x)S(x+1)} \\
&= \frac{f(x+1)[S(x+1) + f(x)] - f(x)S(x+1)}{S(x)S(x+1)} \\
&= \frac{[f(x+1) - f(x)]S(x+1) + f(x)f(x+1)}{S(x)S(x+1)} \\
&= -\eta(x)h(x) + h(x)h(x+1), \text{ from (2).}
\end{aligned}
$$

or

$$
\frac{h(x+1) - h(x)}{h(x)} = h(x+1) - \eta(x). \tag{3}
$$

■

The identity (3) shows that $\frac{\Delta h(x)}{h(x)}$ and $h(x+1) - \eta(x)$ have the same zero, where $\Delta h(x) = h(x+1) - h(x)$. Apart from identifying BT and UBT distributions, the theorem may help in constructing such distributions. Considering equations of the form,

$$
h(x+1) - \eta(x) = a(x),
$$

where $a(x)$ has a zero $x_0 > 0$ may lead to a BT(UBT) law. The method does not require $h(x)$ or $\eta(x)$ but only some functional form $a(x)$ that produces a hazard rate. We give some examples, using simple forms for $a(x)$.

Example 2.1 Let

$$
h(x+1) - \eta(x) = \frac{\alpha x + \beta}{h(x)}. \tag{4}
$$

With the aid of (3), this leads to the recurrence relation,

$$
h(x+1) = \alpha x + \beta + h(x)
$$

and to the solution,

$$
h(x) = \frac{\alpha x(x-1)}{2} + \beta(x-1) + h(0). \tag{5}
$$

We can write (5) in the form,

$$
h(x) = a_1 x^2 + b_1 x + c_1,
$$

which will be a hazard rate if $a_1 > 0$, $c_1 > 0$ and $-2\sqrt{a_1(c_1 - 1)} < b_1 < 0$. Then $h(x)$ has BT shape with change point $[x_0]$, where

$$
x_0 = -\frac{1}{2}\left(1 + \frac{b_1}{a_1}\right).
$$

Table 1: χ^2-test for Example 2.1.

Class	$0-7$	$7-30$	$30-65$	$65-71$	>71
Obs. frequencies	11	8	11	4	16
Exp. frequencies	13	8	9	5	15

The reliability function of the distribution is obtained from (1) as

$$S(x) = \begin{cases} \prod_{t=0}^{x-1}(1-a_1t^2-b_1t-c_1) & : x = 1,2,... \\ 1 & : x = 0 \end{cases} \tag{6}$$

and probability mass function,

$$f(x) = (a_1x^2+b_1x+c_1)\prod_{t=0}^{x-1}(1-a_1t^2-b_1t-c_1).$$

Conversely from (2),

$$\eta(x) = 1-\frac{a_1(x+1)^2+b_1(x+1)+c_1}{a_1x^2+b_1x+c_1}(1-a_1x^2-b_1x-c_1).$$

After some algebra, it can be verified that for the distribution in (6)

$$h(x+1)-\eta(x) = \frac{a_1(2x+1)+b_1}{a_1x^2+b_1x+c} = \frac{\alpha+\beta x}{h(x)},$$

(i.e., the form initially assumed), meaning that the relationship (4) is a characteristic property of (6). We call (6) the quadratic hazard rate distribution. In fact (6) represents a family consisting of the geometric ($a_1 = b_1 = 0$) distribution and the linear hazard rate ($a_1 = 0$, $b_1 > 0$, $c_1 > 0$) distribution with bounded support on $(0,1,...,-\frac{c_1}{b})$, where $-\frac{c_1}{b}$ is a positive integer.

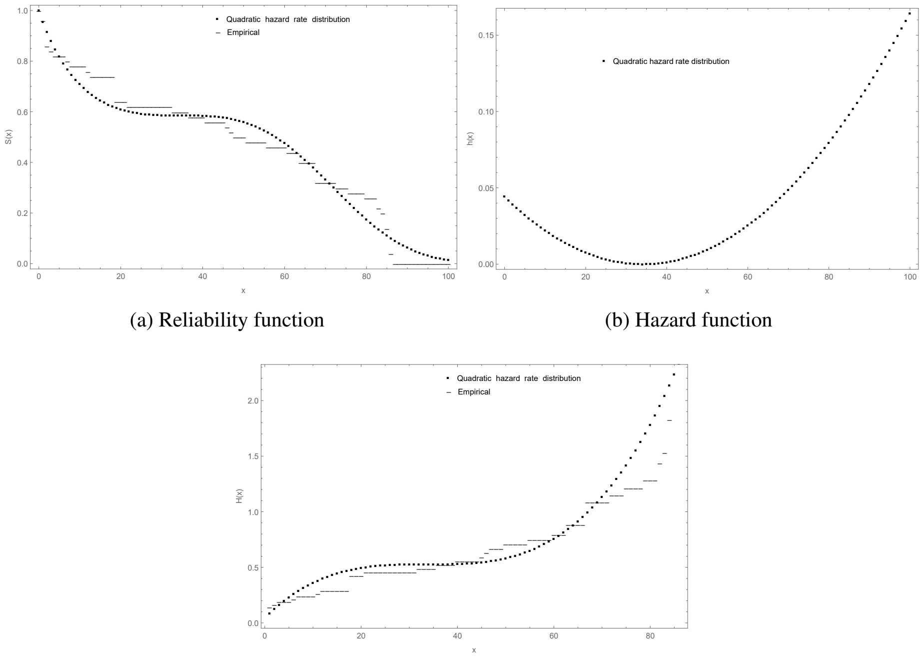
To examine whether the model is useful in practice, we have applied it to the data in Aarset (1987) pertaining to 50 lifetimes of devices by taking the first two observations 0.1 and 0.2 as zeros (the data set is given in Appendix A). The method of least squares is employed to estimate the parameters by minimising

$$L(a_1,b_1,c_1) = \sum_x \left(\sum_{i=0}^{x} a_1i^2+b_1i+c_1-\sum_{i=0}^{x}\frac{\widehat{S}(i)-\widehat{S}(i+1)}{\widehat{S}(i)}\right)^2,$$

where $\widehat{S}(x)$ is the empirical survival function. The estimates obtained were,

$$\widehat{a}_1 = 379\times10^{-7}, \widehat{b}_1 = -259\times10^{-5}, \widehat{c}_1 = 443\times10^{-4},$$

and the error in estimation is $L_{min} = 2.4989$. The model adequacy is checked using χ^2 goodness of fit. The observed and expected frequencies are given in Table 1.



(a) Reliability function

(b) Hazard function

(c) Cumulative hazard function

Figure 1: Survival, hazard and cumulative hazard functions for the data in Example 2.1.

The value of the χ^2 statistic is 1.01 with one degree of freedom and the corresponding p-value is 0.60. Thus the distribution is a good fit and provides a bathtub-shaped hazard function with change point $x_0 = 33$. The plots of the reliability function, hazard function and cumulative hazard function are exhibited in Figures 1a–1c. From the graphs it is clear that the model fits the data well.

Example 2.2 Consider the identity,

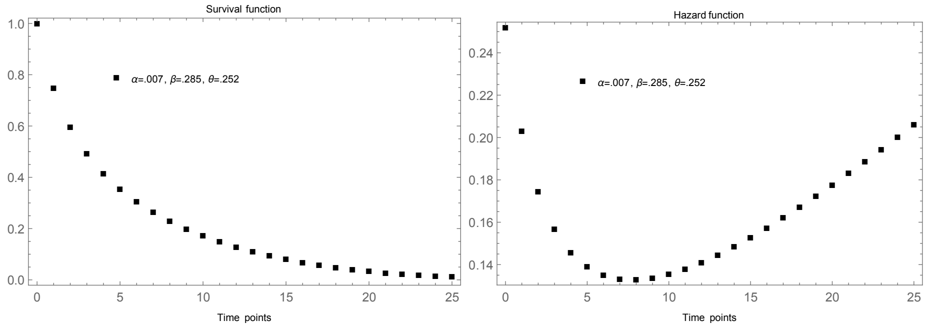
$$h(x+1) - \eta(x) = \left(\alpha - \frac{\frac{\theta\beta}{(1+\beta x)(1+\beta(x+1))}}{h(x)} \right).$$

Now,

$$h(x+1) - h(x) = \alpha + \frac{\theta}{1+\beta(x+1)} - \frac{\theta}{1+\beta x},$$

leaving the solution,

$$h(x) = \alpha x + \frac{\theta}{1+\beta x}, \theta = h(0). \tag{7}$$



(a) Reliability function

(b) Hazard function

Figure 2: Survival and hazard functions for the model in Example 2.2.

In order for (7) to be a hazard rate, one should have α, $\beta > 0$, $0 < \theta < 1$. The reliability function is

$$S(x) = \begin{cases} \prod_{t=0}^{x-1} \left(1 - \alpha t - \frac{\theta}{1+\beta t}\right) & : x = 1, 2, 3 \ldots \\ 1 & : x = 0. \end{cases} \tag{8}$$

Note that $h(x)$ is BT when $0 < \alpha < \frac{\theta\beta}{1+\beta}$. The form of the hazard rate is similar to the hazard rate in the continuous case obtained by Hjorth (1980). However, the reliability function (8) is not the discretised version of the Hjorth model. The expression (7) is the sum of the hazard rates of the linear hazard rate distribution and the Waring distribution. It is known that by taking the sum of two hazard rates, one of which is decreasing and the other increasing, we may obtain a BT hazard rate. This is also suggested as a method for deriving a new BT model. The above example can also be seen in this context. There are several continuous distributions based on hazard rates having such a structure; see, for example, Murthy, Xie and Jiang (2004), Jaisingh, Kolarik and Dey (1987), Canfield and Borgman (1975), Xie and Lai (1996), Jiang and Murthy (1997), Usgaonkar and Mariappan (2009) and Wang (2000). The method discussed in this section can be considered in these cases as well by appropriately choosing $a(x)$. From (7) it is easy to see that the change point x_0 is the solution of the quadratic equation

$$\beta x^2 + \alpha x + \theta,$$

provided that $4\beta\theta < \alpha^2 < 4\beta\theta - \beta^2 - \beta$ and $x_0 > 0$. The parameters of the model are estimated by minimising the discrepancy

$$\sum \left(\alpha x + \frac{\theta}{1+\beta x} - \frac{\widehat{S}(x) - \widehat{S}(x+1)}{\widehat{S}(x)} \right)^2$$

between the model and the empirical hazard rates. Since $\theta = h(0)$, we take it as the observed value of $h(0)$. Thus the only parameters to be estimated are α and β. The hazard rate function and the reliability function can be seen in Figures 2a–2b.

3. Discretising Continuous Bathtub Distribution

Let Y be a continuous lifetime random variable with reliability function $\bar{F}(x) = P[Y \geq x]$. If time is recorded at unit intervals, the discrete random variable $X = [Y]$, the integer part of Y, has the reliability function $S(x) = \bar{F}(x)$, $x = 0, 1, 2...$ and probability mass function,

$$f(x) = S(x) - S(x+1).$$

When Y has a bathtub hazard rate, it may turn out that X also has a BT hazard rate. The reliability functions $S_1(x)$ through $S_4(x)$ discussed earlier were obtained in this way. We shall further illustrate this method with two examples, one of which renders BT and the other UBT.

Example 3.1 One of the earliest bathtub models introduced by Bain (1974) and Bain and Englehardt (1991) was the quadratic hazard rate model with

$$\bar{F}(x) = \exp\left[-ax - \frac{bx^2}{2} - \frac{cx^3}{3}\right], x > 0, c > 0, b \geq -(2ac)^{\frac{1}{2}}.$$

The reliability function and probability mass function of the corresponding discrete model is given by

$$S(x) = q^{\left(ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right)}, q = e^{-1} \tag{9}$$

and

$$f(x) = q^{\left(ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right)} \left[1 - q^{a + \frac{b}{2}(2x+1) + \frac{c}{3}(3x^2+3x+1)}\right].$$

Subsequently, we have that

$$h(x) = 1 - q^{a + \frac{b}{2}(2x+1) + \frac{c}{3}(3x^2+3x+1)}.$$

The model introduced in (9) will be called a discretised quadratic hazard model and it will be denoted by DQHM(a, b, c). We study the model in detail in Section 6, where we show that DQHM possesses a BT shaped hazard rate for specified values of the parameters.

Example 3.2 The log logistic distribution (Gupta, Akman and Lvin, 1999) of a continuous random variable Y is specified by the reliability function

$$\bar{F}(x) = P[Y > x] = \frac{1}{1 + cx^{\alpha}}, x \geq 0, \alpha > 0.$$

It is known that this distribution has a decreasing (UBT shaped) hazard function, when $\alpha \leq (>) 1$. In the UBT case, the change point is given by,

$$x_0 = \left(\frac{\alpha - 1}{c}\right)^{\frac{1}{\alpha}}.$$

The application of the distribution in analysing survival data has been pointed out by several authors. We refer to Gupta et al. (1999) and their references for details. The integer part X of Y has reliability function,

$$S(x) = \frac{1}{1 + cx^{\alpha}}, x = 0, 1, 2, ..., \alpha > 0,$$

probability mass function

$$f(x) = \frac{1}{1 + cx^{\alpha}} - \frac{1}{1 + c(x+1)^{\alpha}}$$

Table 2: χ^2-test for leukaemia data.

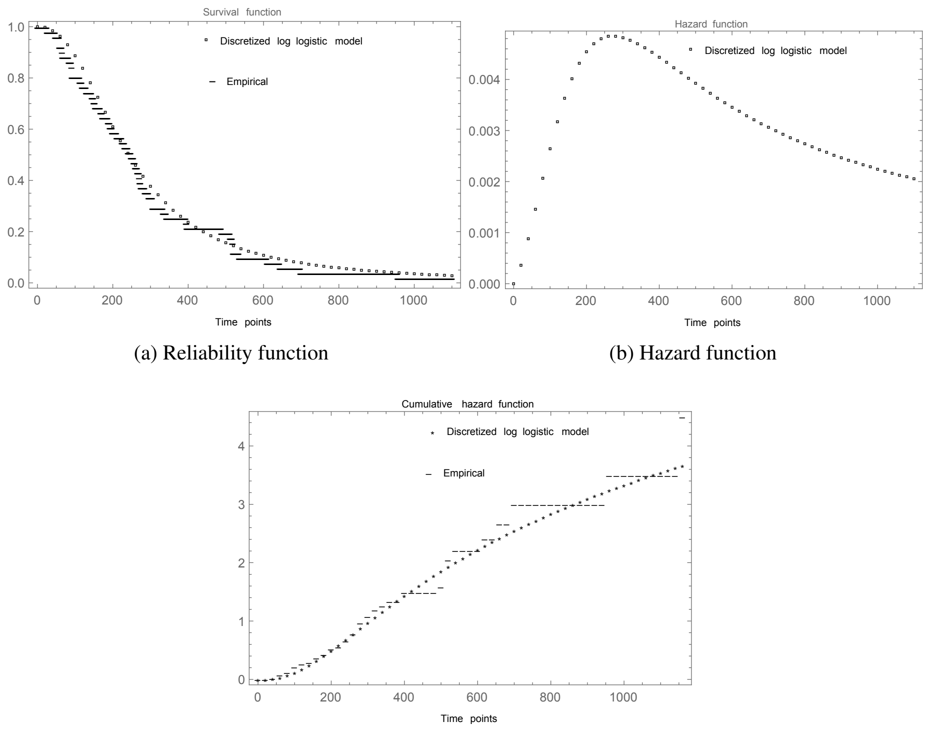
Class	$0-100$	$100-150$	$151-200$	$201-250$	$251-300$	$301-400$	>400
Obs. frequencies	6	7	7	7	5	7	12
Exp. frequencies	10	5	6	5	8	6	11

and hazard function

$$h(x) = 1 - \frac{1+cx^{\alpha}}{1+c(x+1)^{\alpha}}.$$

To ascertain the usefulness of the model, we apply it to the data on the times from remission to relapse of 84 patients with acute non-lymphoblastic leukaemia reported in Glucksberg et al. (1981). For the present analysis, the censored observations are omitted and the rest of the 51 observations are only utilised (the data set is given in Appendix A). We minimise the squared distance between $S(x)$ and $\widehat{S}(x)$ to estimate the parameters of the model. This gives the estimates

$$\widehat{\alpha} = 2.33009 \text{ and } \widehat{c} = 2.78614 \times 10^{-6}$$



(a) Reliability function

(b) Hazard function

(c) Cumulative hazard function

Figure 3: Survival, hazard and cumulative hazard functions for the leukaemia data in Example 3.2.

and the error between the fitted values and observed survival probabilities is 0.573 for the above $\widehat{\alpha}$ and $\widehat{c}$. The model adequacy is checked through the χ^2-test. The observed and expected frequencies are shown in Table 2 and the graphs of the reliability, hazard rate and cumulative hazard rate functions are given as Figures 3a–3c.

The χ^2 value of 5.97 at 4 degrees of freedom yields a p-value of 0.20.

4. Modifying Decreasing Hazard Rate Functions

A third method that may result in BT distributions is to consider

$$h^*(x) = \frac{h(x)}{S(x)},\, h(x) \leq S(x), \tag{10}$$

where $h(x)$ is a decreasing hazard rate with reliability function $S(x)$. Under the given conditions, $0 \leq h^*(x) \leq 1$ and

$$\sum_{x=0}^{\infty} h^*(x) = \sum_{x=0}^{\infty} \frac{h(x)}{S(x)} \geq \sum_{x=0}^{\infty} h(x) = \infty,$$

so that $h^*(x)$ is a hazard rate with reliability function $S^*(x)$. Now consider

$$\begin{aligned} h^*(x+1) - h^*(x) &= \frac{h(x+1)S(x) - h(x)S(x+1)}{S(x)S(x+1)} \\ &= \frac{h(x+1)}{S(x+1)} - \frac{h(x)}{S(x)}. \end{aligned}$$

For $h^*(x)$ to be BT, the right of above expression must be zero for a unique $x_0 > 0$. But,

$$\frac{h(x+1)}{S(x+1)} = \frac{h(x)}{S(x)}$$

implies

$$\frac{h(x+1)}{h(x)} = \frac{S(x+1)}{S(x)}$$

or

$$\frac{h(x+1)}{h(x)} = 1 - h(x).$$

Thus for $h^*(x)$ to be BT or UBT,

$$h(x+1) = h(x)(1-h(x))$$

must have a unique solution $x_0 > 0$. The idea behind the modification in (10) is that initially $S(x)$ has values close to unity to keep the decreasing nature of $h(x)$ and hence that of $h^*(x)$. But as x increases, $S(x)$ becomes closer to zero to increase the value of $h(x)$ that may transform $h^*(x)$ to an increasing function, so that the overall shape of $h^*(x)$ may be BT. If $h^*(x)$ does not produce a BT, then the process can be repeated with $h^{**}(x) = \dfrac{h^*(x)}{S^*(x)}$, provided $h^*(x) \leq S^*(x)$ and so on. We give two examples that illustrate how the method works in practice.

Example 4.1 Let X follow the Waring (Nair, Sankaran and Preeth, 2012) distribution,

$$S(x) = \frac{(b)_x}{(a)_x},\ x = 0, 1, 2...;\ a > b,$$

where $(a)_x = a(a+1)...(a+x-1)$ is the Pochammer's symbol and $(a)_0 = 1$. Then,

$$h(x) = \frac{a-b}{a+x},$$

which is clearly decreasing, and

$$h^*(x) = \frac{(a-b)(a)_x}{(a+x)(b)_x}.$$

By virtue of the Waring expansion,

$$\frac{1}{(x-a)} = \frac{1}{x} + \frac{a}{x(x+1)} + \frac{a(a+1)}{x(x+1)(x+2)} + ...,$$

and we can write

$$S(x) = \frac{(b)_x}{(a)_x} = (a-b)\frac{(b)_x}{(a)_x}\left[\frac{1}{a+x} + \frac{b+x}{(a+x)(a+x+1)} + ...\right].$$

From this, it can be seen that $h(x) \leq S(x)$. Also,

$$h(x+1) - h(x)(1-h(x)) = 0$$

leads to

$$a^2 + ax - bx - x - ab - b = 0.$$

The unique solution to this equation is

$$x_0 = \frac{ab + b - a^2}{a-b-1},$$

which will give a change point provided $(a-1) < b < \frac{a^2}{a+1}$. As an illustration, taking $a = 1.32$, $b = 0.46$, we have $x_0 = 4.8$. Some simple numerical calculations show that $h^*(x)$ is increasing in $[0, b)$ and decreasing in $[b, \infty)$ confirming its UBT property. The reliability function $S^*(x)$ is derived from (1).

Example 4.2 A good share of continuous bathtub distributions are related to the Weibull distribution as can be seen from Chapter 5 of Lai and Xie (2006). In this example, we apply the above method to generate a BT model from the discretised Weibull distribution, i.e.,

$$S(x) = q^{x^\beta},\ x = 0, 1, 2...,\ 0 < q < 1,\ \beta > 0$$

of Nakagawa and Osaki (1975). In this case,

$$h(x) = 1 - \frac{q^{(x+1)^\beta}}{q^{x^\beta}},$$

so that

$$h^*(x) = \frac{q^{x^\beta} - q^{(x+1)^\beta}}{q^{2x^\beta}}$$

and $h(x) \leq S(x)$. From Figure 4 representing the graph of $h^*(x)$, it is seen that $h^*(x)$ can be BT.

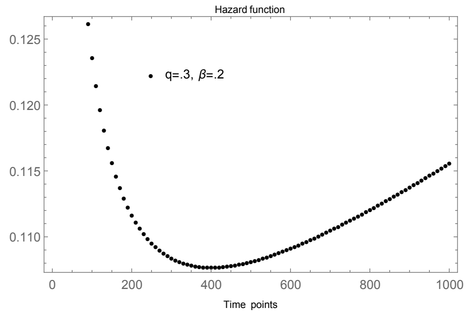


Figure 4: Hazard function for the model in Example 4.2.

5. Other Methods

In this section, we discuss certain methods borrowed from the continuous case. The mixture of a distribution with increasing hazard rate and a distribution with decreasing hazard rate may produce a BT distribution. Let $f_1(x)(S_1(x))$ and $f_2(x)(S_2(x))$ be the probability mass(reliability) functions of two discrete lifetimes X_1 and X_2. Then the two-component mixture of $f_1(x)$ and $f_2(x)$,

$$f(x) = \alpha f_1(x) + (1-\alpha) f_2(x),\, 0 \leq \alpha \leq 1$$

has a hazard rate of the form

$$h(x) = p(x)h_1(x) + (1-p(x))h_2(x), \tag{11}$$

where $h_1(x)$ and $h_2(x)$ are the hazard rate functions of X_1 and X_2 and

$$p(x) = \frac{\alpha S_1(x)}{\alpha S_1(x) + (1-\alpha)S_2(x)}.$$

Although the expression (11) looks compact, it is difficult to prove analytically that $h(x)$ has a maximum or minimum.

A strictly convex function which satisfies $0 \leq h(x) \leq 1$ and $\sum_{t=0}^{\infty} h(t) = \infty$ for non-negative integer values can be a candidate hazard rate function that is BT shaped.

Example 5.1 The function

$$h(x) = 1 - e^{-(ax^2+bx+c)},\, b > -a,\, a > 0 \tag{12}$$

satisfies the above conditions. The parameters are estimated by the regression of $-\log(1-\widehat{h}(x))$, where $\widehat{h}(x) = \dfrac{\widehat{S}(x) - \widehat{S}(x+1)}{\widehat{S}(x)}$ is the empirical hazard rate, on a quadratic function. This method was applied to the analysis of data in Aarset (1987) pertaining to 50 lifetimes of devices by taking the first two observations 0.1 and 0.2 as zeros to obtain the estimates

$$\widehat{a} = 227.975 \times 10^{-7},\, \widehat{b} = -156.645 \times 10^{-5},\, \widehat{c} = 326.186 \times 10^{-4}$$

Table 3: Observed and expected frequencies for Aarset data.

Class	$0-4$	$5-18$	$19-50$	$51-67$	$68-84$	>84
Observed	9	9	8	8	9	7
Expected	7	10	7	8	10	8

The sum of squares of the errors between the model and empirical values is 0.041. Applying the χ^2-test, we have the observed and expected frequencies as in Table 3.

The χ^2 value of 1.03 at 2 degrees of freedom gives a p-value of 0.59. The change point is

$$x_0 = \left[-\frac{(b+a)}{2a}\right] = 33,$$

the integer part of x_0. See Figures 5a–5c for the reliability, hazard rate and cumulative hazard rate functions. From (1) and (12), we arrive at a nice form for the reliability function as

$$S(x) = q^{\frac{ax^3}{3}+\frac{a-b}{2}x^2+\frac{(a-3b-6c)}{6}x}.$$

The quadratic hazard rate family of Example 3.1 is another distribution that obeys the above criterion.

Another useful method is to consider series systems in which the hazard rate of the system is the sum of the hazard rates of the components. This method was already mentioned in connection with Example 2.2.

6. Discretised Quadratic Hazard Model

The discretised quadratic hazard model (9), introduced in Section 3 deserves a separate study because of its interesting reliability properties. In this section, we study the reliability properties of DQHM and propose a two stage procedure for estimating the parameters. A real data set has been analysed using this estimation procedure and we can see that the model performs well.

The model,

$$h(x+1)-h(x) = q^{a+\frac{b}{2}(2x+1)+\frac{c}{3}(3x^2+3x+1)}\left(1-q^{b+2c(x+1)}\right),$$

has a unique zero

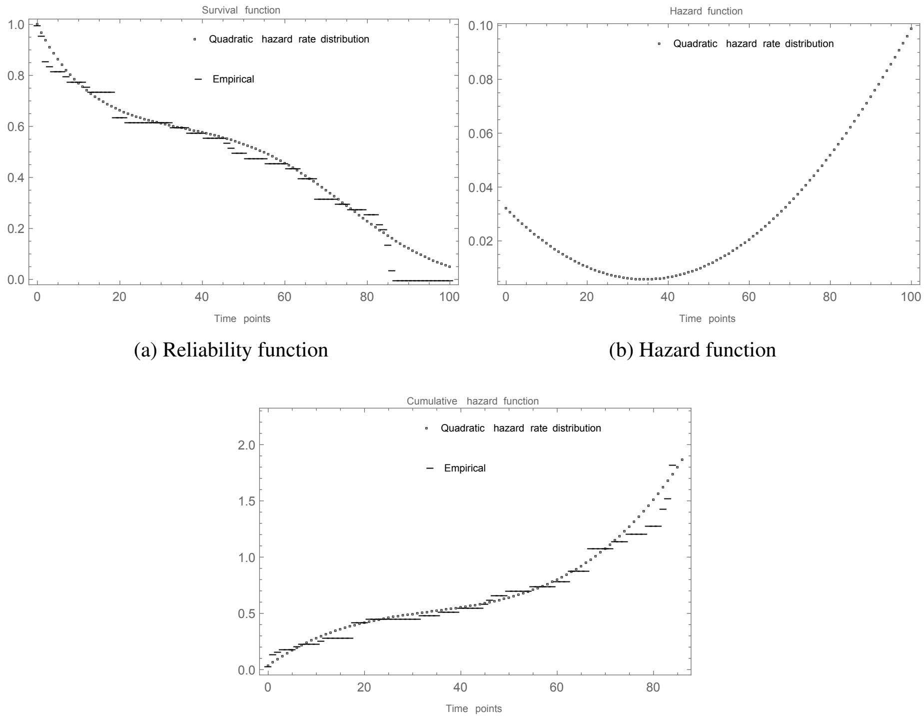
$$[x_0] = \left[-\frac{b}{2c}-1\right], b<0, c>0,$$

where $x_0>0$ and $[x_0]$ is the integer part of x_0. For x_0 to be non negative, we need $-b>2c$. Further,

$$h(x_0+1)-h(x_0) = 1-q>0,$$

and

$$h(x_0)-h(x_0-1) = 1-q^{-2c}<0,$$



(a) Reliability function

(b) Hazard function

(c) Cumulative hazard function

Figure 5: Survival, hazard and cumulative hazard functions for the model in Example 5.1.

showing that $h(x)$ is decreasing in $[0,x_0)$ and increasing in $[x_0,\infty)$ yielding a BT shape. Thus the hazard rate function is BT when $-b > 2c$ and is increasing for $b > 0$. The DQHM(a,b,c) has a non-zero hazard rate at the point 0, which is not common.

The following particular cases are applicable for DQHM(a,b,c).

- When $b = c = 0$, the model reduces to the geometric distribution with parameter $\theta = q^a$.
- When $c = 0$ and $b > 0$, it has the hazard rate function

$$h(x) = 1 - q^{-(a+\frac{b}{2}+bx)},$$

  which is increasing.

- When $a = c = 0$, we have the discretised version of the Rayleigh distribution.

Theorem 2 Consider a series system consisting of n components. Let the component lifetimes be independently distributed as DQHM(a_i,b_i,c_i), $i = 1,2,...,n$. Then the system lifetime is distributed as DQHM$(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} b_i, \sum_{i=1}^{n} c_i)$.

The proof is direct.

6.1. Residual life

The concept of residual life plays an important role in reliability analysis. The residual life random variable X_t is defined as

$$X_t = X - t | X \geq t, t = 1, ...$$

The survival function corresponding to the residual life X_t is defined as

$$S_t(x) = \frac{S(x+t)}{S(t)}, x = 1, 2....$$

The following theorem gives the closure property of the residual life random variable of DQHM(a,b,c).

Theorem 3 For DQHM(a,b,c), the residual life variable X_t is distributed as DQHM (a_1, b_1, c_1) with survival function

$$S_t(x) = q^{a_1 x + \frac{b_1}{2} x^2 + \frac{c_1}{3} x^3},$$

where $a_1 = a + bt + ct^2$, $b_1 = (b + 2ct)$ and $c_1 = c$.

The proof of the theorem follows directly from the definition of $S_t(x)$.

6.2. Transformations

In this section, we study the behaviour of DQHM distribution under scale transformation. The following theorem shows the closure of DQHM under change of scale; the proof is direct.

Theorem 4 Let X be a non-negative integer valued random variable and $k > 0$ be a constant. Then $Y = kX$ is distributed as DQHM(a_1, b_1, c_1) if and only if X follows DQHM(a,b,c), where $a_1 = \frac{a}{k}$, $b_1 = \frac{b}{k^2}$ and $c_1 = \frac{c}{k^3}$.

6.3. Estimation of parameters

Let $X_1, X_2 ... X_n$ be a random sample from (9). We apply the maximum likelihood procedure for estimating the parameters. In the present set up, the likelihood is very complicated and there is a possibility of multiple roots for the score equation. The convergence of the estimates depends on the initial value we give. So we consider a two stage estimation procedure, which consists of estimating the initial values of the parameters by least square fit and then maximising the likelihood with these estimates as starting points. Based on the random sample from the DQHM(a,b,c), the log likelihood is given by

$$\begin{aligned} l[\underline{x}, a, b, c] &= \sum_{i=1}^{n} \log[e^{-ax_i - \frac{bx_i^2}{2} - \frac{cx_i^3}{3}} (1 - e^{-(a + \frac{b}{2} + \frac{c}{3} + (b+c)x_i + cx_i^2)})] \\ &= \sum_{i=1}^{n} (-ax_i - \frac{bx_i^2}{2} - \frac{cx_i^3}{3}) + \log(1 - e^{-(a + \frac{b}{2} + \frac{c}{3} + (b+c)x_i + cx_i^2)}) \end{aligned}$$

The score equations are

$$
\begin{aligned}
\frac{\delta}{\delta a} l[\underline{x},a,b,c] &= \sum_{i=1}^{n} \left[\frac{x_i \left(-e^{a+x_i(b+c)+\frac{b}{2}+cx_i^2+\frac{c}{3}} \right) + x_i + 1}{e^{a+x_i(b+c)+\frac{b}{2}+cx_i^2+\frac{c}{3}} - 1} \right] = 0, \\
\frac{\delta}{\delta b} l[\underline{x},a,b,c] &= \sum_{i=1}^{n} \frac{x_i^2 \left(-\left(e^{a+x_i(b+c)+\frac{b}{2}+cx_i^2+\frac{c}{3}} - 1 \right) \right) + 2x_i + 1}{2\left(e^{a+x_i(b+c)+\frac{b}{2}+cx_i^2+\frac{c}{3}} - 1 \right)} = 0, \\
\frac{\delta}{\delta c} l[\underline{x},a,b,c] &= \sum_{i=1}^{n} \frac{x_i^3 \left(-\left(e^{a+x_i(b+c)+\frac{b}{2}+cx_i^2+\frac{c}{3}} - 1 \right) \right) + 3x_i^2 + 3x_i + 1}{3\left(e^{a+x_i(b+c)+\frac{b}{2}+cx_i^2+\frac{c}{3}} - 1 \right)} = 0.
\end{aligned}
$$

The second order derivatives are given by

$$\frac{\delta^2}{\delta a^2} l[\underline{x},a,b,c] = \sum_{i=1}^{n} \frac{1}{2 - 2\cosh\left(a + b\left(x_i + \frac{1}{2} \right) + c\left(x_i^2 + x_i + \frac{1}{3} \right) \right)}, \tag{13}$$

$$\frac{\delta^2}{\delta b^2} l[\underline{x},a,b,c] = \sum_{i=1}^{n} -\frac{(2x_i+1)^2}{8\left(\cosh\left(a + b\left(x_i + \frac{1}{2} \right) + c\left(x_i^2 + x_i + \frac{1}{3} \right) \right) - 1 \right)}, \tag{14}$$

$$\frac{\delta^2}{\delta c^2} l[\underline{x},a,b,c] = \sum_{i=1}^{n} -\frac{(3x_i(x_i+1)+1)^2}{18\left(\cosh\left(a + b\left(x_i + \frac{1}{2} \right) + c\left(x_i^2 + x_i + \frac{1}{3} \right) \right) - 1 \right)}, \tag{15}$$

$$\frac{\delta^2}{\delta ab} l[\underline{x},a,b,c] = \sum_{i=1}^{n} -\frac{2x_i+1}{4\left(\cosh\left(a + b\left(x_i + \frac{1}{2} \right) + c\left(x_i^2 + x_i + \frac{1}{3} \right) \right) - 1 \right)}, \tag{16}$$

$$\frac{\delta^2}{\delta ac} l[\underline{x},a,b,c] = \sum_{i=1}^{n} -\frac{3x_i(x_i+1)+1}{6\left(\cosh\left(a + b\left(x_i + \frac{1}{2} \right) + c\left(x_i^2 + x_i + \frac{1}{3} \right) \right) - 1 \right)}, \tag{17}$$

$$\frac{\delta^2}{\delta bc} l[\underline{x},a,b,c] = \sum_{i=1}^{n} -\frac{(2x_i+1)(3x_i(x_i+1)+1)}{12\left(\cosh\left(a + b\left(x_i + \frac{1}{2} \right) + c\left(x_i^2 + x_i + \frac{1}{3} \right) \right) - 1 \right)}. \tag{18}$$

We can see that the score equations are non-linear in a,b and c. We need to use numerical methods to solve them. As mentioned before, we need appropriate initial values to use numerical methods effectively. To obtain these initial values, we proceed as follows. Let $\widehat{S}(x)$ be the empirical survival function calculated from the sample. We propose a linear regression model

$$-\,log(\widehat{S}(x_i)) = ax_i + \frac{bx_i^2}{2} + \frac{cx_i^3}{3} + \varepsilon_i,\ i = 1,2...n, \tag{19}$$

where ε_i's are independent and identically distributed random variables with mean 0 and variance σ^2.

The model in (19) can be rewritten in matrix form as

$$y = M\theta + \varepsilon,$$

where $y=[-log(\widehat{S}(x_i))]$, $i=1,...n$, $\theta=[a,b,c]$, and $\varepsilon=[\varepsilon_1,...,\varepsilon_n]$. The design matrix corresponding to the above model is given by

$$M=\begin{bmatrix} x_1 & \frac{x_1^2}{2} & \frac{x_1^3}{3} \\ x_2 & \frac{x_2^2}{2} & \frac{x_2^3}{3} \\ \cdots & \cdots & \cdots \\ x_n & \frac{x_n^2}{2} & \frac{x_n^3}{3} \end{bmatrix}.$$

An estimate of θ is obtained by using the ordinary least square method. The solution is given by

$$\widehat{\theta}=[\widehat{a},\widehat{b},\widehat{c}]=(M'M)^{-1}M'y.$$

We use these estimates as initial values for the maximisation of the log likelihood. It is easy to see that the probability mass function satisfies the regularity condition given by Cramér (1999). Thus by Cramér-Huzurbazar theorem (see Lehmann and Casella, 1998), we can see that $\widehat{\theta}$ is consistent and $\sqrt{n}(\widehat{\theta}-\theta)$ is asymptotically normal with mean vector $\underline{0}$ and dispersion matrix $\frac{1}{\sqrt{n}}I^{-1}(\theta)$, where $I(\theta)$ is the Fisher information matrix. From (13)–(18) we can evaluate the observed Fisher information matrix numerically, which gives an estimate of $I(\theta)$. We now illustrate the method with a real data set. We compare the model performance with other existing models.

Example 6.1 We consider a dataset consisting of the lifetimes of 18 electronic components (the data set is given in Appendix A), reported in Wang (2000), which was recently analysed by Almalki and Nadarajah (2014) using the discretised reduced modified Weibull(DRMW) distribution. To obtain the least square estimates, we form the design matrix as

$$M=\begin{bmatrix} x_1 & \frac{x_1^2}{2} & \frac{x_1^3}{3} \\ x_2 & \frac{x_2^2}{2} & \frac{x_2^3}{3} \\ \cdots & \cdots & \cdots \\ x_{18} & \frac{x_{18}^2}{2} & \frac{x_{18}^3}{3} \end{bmatrix}$$

and propose the model,

$$-\log(\widehat{S}(x_i))=ax_i+\frac{bx_i^2}{2}+\frac{cx_i^3}{3}+\varepsilon_i,\ i=1,2...18,$$

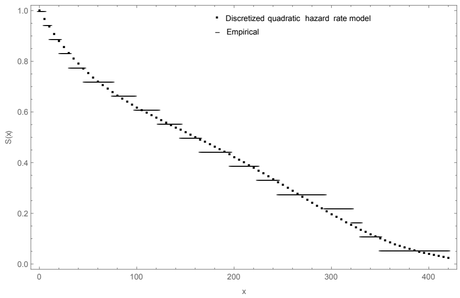
where $\widehat{S}(x)$ is the empirical survival function.

The least square estimates are

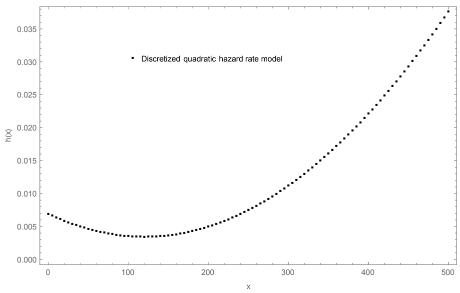
$$\widehat{a}=532.272\times10^{-5}, \widehat{b}=-303.786\times10^{-7} \text{ and } \widehat{c}=1.3723\times10^{-7}.$$

Using these as initial estimates, the log likelihood is numerically maximised. The maximum of l is obtained as

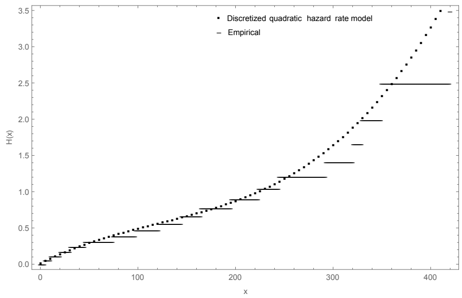
$$l_{max}=-108.213$$



(a) Survival function



(b) Hazard function



(c) Cumulative hazard function

Figure 6: Survival, hazard and cumulative hazard functions for the data in Example 6.1.

for the values

$$\widehat{a} = 695.067 \times 10^{-5}, \widehat{b} = -585.678 \times 10^{-7} \text{ and } \widehat{c} = 2.4217 \times 10^{-7}.$$

The survival function, hazard rate function and cumulative hazard rate function are plotted in Figures 6a–6c.

Now, to compare the performance of DQHM with the existing models, we calculate the Kolmogorov Smirnov distance, Akaike Information Criterion(AIC), Bayesian Information Criterion (BIC) and Consistent Akaike Information Criterion (CAIC).

We have,

$$AIC = 2k - 2l[\widehat{\theta}, \underline{x}],$$

where k is the dimension of the vector θ and $l[\widehat{\theta}, \underline{x}]$ is the log likelihood at $\widehat{\theta}$.

Also,

$$BIC = k \log n - 2l[\widehat{\theta}, \underline{x}]$$

and

$$CAIC = AIC + \frac{2k(k+1)}{n-k-1}.$$

Table 4 provides the values of these measures of model adequacy. From Table 4, we see that the DQHM model outperforms the other four models, namely the discrete reduced modified

Table 4: Model adequacy of the data in Example 6.1.

Model	AIC	BIC	CAIC	K-S
DRMW	223.9	226.5	225.6	.084
DMW	225.6	228.3	227.3	.092
DAddW	227.9	231.4	230.9	.099
DW	226.1	227.9	226.9	.137
DQHM	**222.426**	**225.097**	**224.14**	**0.0702**

Weibull(DRMW) due to Almalki and Nadarajah (2014), the discrete modified Weibull(DMW) due to Noughabi et al. (2011), the discrete additive Weibull (DAddW) due to Bebbington et al. (2012) and the discrete Weibull (DW) due to Nakagawa and Osaki (1975).

7. Conclusion

In the present paper we have discussed various methods for the construction of discrete bathtub distributions. We have provided examples in which the models were applied to real data and we have studied the properties of the discretised quadratic hazard model in detail. These supplement the existing list of BT models in literature. Only a few methods that are analogues of the continuous case have been included in the discussion. Some interesting methods such as those based on total time on test transforms, functions of random variables, additive hazard rate models, etc. could not be considered here. The definitions, concepts and technical results needed to use them for BT models have not yet been developed in the discrete case. Some attempts are being made in this direction and will be reported elsewhere.

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Appendix A

Table 5: Data in Aarset (1987) pertaining to 50 lifetimes of devices.

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12	18	18	18	18
18	21	32	36	40	45	46	47	50	55	60	63	63	67	67	67	67
72	75	79	82	82	83	84	84	84	85	85	85	85	85	86	86	

Table 6: Data in Glucksberg et al. (1981) on the times from remission to relapse of patients with acute non-lymphoblastic leukaemia. Out of 84 observations, only 51 uncensored observations are included here.

24	46	57	57	64	65	82	89	90	90	111	117	128
43	148	152	166	171	186	191	197	209	223	230	239	247
254	258	264	269	270	273	284	294	304	304	332	341	393
395	487	510	516	518	518	534	608	642	697	955	1160	

Table 7: Dataset consisting of the lifetimes of 18 electronic components, reported in Wang (2000)

5	11	21	31	46	75	98	122	145
165	196	224	245	293	321	330	350	420

Appendix B

We used Wolfram Mathematica® 10 program from Wolfram Research, Inc. for the mathematical computations. For numerical maximisation, we used **NMaximize** function, which gives a global maximum, and **FindMaximum** function, which gives a local maximum when initial values are available. For computing the empirical survival function, we used **SurvivalModelFit** function for non-censored data. The design matrix for the two stage estimation procedure was computed using **DesignMatrix** function and the linear regression analysis was carried out using **LinearModelFit** function. For more details of these functions, see the help at reference.wolfram.com/

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