ON FIVE-PARAMETER BURR XII DISTRIBUTION: PROPERTIES AND APPLICATIONS

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Abstract: A new five-parameter distribution is defined and studied. The new model is referred to as the Kumaraswamy exponentiated Burr XII distribution. The new distribution includes eleven special models, namely, the Kumaraswamy exponentiated log-logistic, Kumaraswamy exponentiated Lomax, Kumaraswamy Burr XII and Kumaraswamy log-logistic distributions among others. Various structural properties of this model including moments, moment generating function, incomplete moments, order statistics and Rényi entropy are derived. The maximum likelihood method is used for estimating the model parameters. We illustrate the improved performance of the proposed distribution over other distributions available in the literature when modelling two real data sets.

1. Introduction

The Burr-XII (BXII) distribution, which was originally derived by Burr (1942) and since then has received more attention by statisticians due to its broad applications in different fields, including reliability, failure time modelling and acceptance sampling plans. Shao, Wong, Xia and Ip (2004) studied the models for the extended three parameter Burr type XII distribution and used this distribution to model extreme events with application to flood frequency. The BXII model is an important distribution because it includes some well-known sub-models namely: Pareto II (Lomax), log-logistic, compound Weibull Gamma and Weibull exponential distributions.

Recently, many authors constructed generalisations of the BXII distribution. For example, Paranaíba, Ortega, Cordeiro and Pescim (2011) proposed the beta Burr XII (BBXII), Paranaíba, Ortega, Cordeiro

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and de Pascoa (2013) introduced the Kumaraswamy Burr XII (KBXII), Antonio, da Silva and Cordeiro (2014) proposed the McDonald Burr XII and Mead (2014) introduced the beta exponentiated Burr XII (BEBXII) distributions.

The cumulative distribution function (cdf) of the exponentiated Burr XII (EBXII) distribution is given (for $x \geq 0$) by

$$G(x;c,k,\beta) = \left[1-(1+x^c)^{-k}\right]^{\beta}, \tag{1}$$

where $c > 0$, $k > 0$ and $\beta > 0$.

The corresponding probability density function (pdf) is given by

$$g(x;c,k,\beta) = ck\beta x^{c-1}(1+x^c)^{-(k+1)}\left[1-(1+x^c)^{-k}\right]^{\beta-1}. \tag{2}$$

The aim of this paper is to propose and study a new lifetime model called the Kumaraswamy exponentiated Burr XII (KEBXII for short) distribution. The main feature of this model is that the two additional shape parameters inserted in (2) can give greater flexibility in the form of the new density. By using the Kumaraswamy-generalised (K-G) family of distributions pioneered by Cordeiro and de Castro (2011), we construct the five-parameter KEBXII model. We provide some of its mathematical properties with the hope that it can serve as an alternative model to other existing lifetime models in the literature for analysing and modelling real data in economics, reliability, engineering and other areas of research.

Let $G(x)$ be the baseline cdf, Cordeiro and de Castro (2011) defined the K-G family by the cdf and pdf given by

$$F(x;a,b) = 1-\{1-G(x)^a\}^b \tag{3}$$

and

$$f(x;a,b) = a\,b\,g(x)\,G(x)^{a-1}\{1-G(x)^a\}^{b-1}, \tag{4}$$

respectively, where $g(x) = dG(x)/dx$ and $a > 0$ and $b > 0$ are two additional shape parameters. For $a = b = 1$, we obtain the baseline distribution. The additional parameters a and b aim to govern skewness and tail weight of the generated model. An attractive feature of this family is that a and b can afford greater control over the weights in both tails and in the centre of the distribution.

Many authors used the K-G family to construct new generalisations of some well-known distributions. For example, Cordeiro, Ortega and Nadarajah (2010) defined the Kumaraswamy Weibull, de Castro, Ortega and Cordeiro (2011) proposed the Kumaraswamy generalised gamma, Cordeiro, Pescim and Ortega (2012) introduced the Kumaraswamy generalised half-normal, Bourguignon, Silva, Zea and Cordeiro (2013) proposed the Kumaraswamy Pareto and Afify, Cordeiro, Butt, Ortega and Suzuki (2016) proposed the Kumaraswamy complementary Weibull geometric distributions.

The rest of the paper is outlined as follows. In Section 2, we define the KEBXII distribution and give some plots for its pdf and hazard rate function (hrf). In Section 3, a useful mixture representations for its pdf and cdf and some mathematical properties of the KEBXII distribution including, ordinary and incomplete moments, mean residual life, mean waiting time, quantile and generating functions, Lorenz, Bonferroni and Zenga curves, order statistics and Rényi entropy are derived. The maximum likelihood estimates (MLEs) of the model parameters are obtained in Section 4. The KE-BXII distribution is applied to two real data sets to illustrate its potential in Section 5. Finally, in Section 6, we provide some concluding remarks.

2. The KEBXII Distribution

By substituting (1) in (3), the cdf of the KEBXII distribution is given by (for $x > 0$)

$$F(x;\varphi) = 1 - \left\{1 - \left[1 - (1+x^c)^{-k}\right]^{a\beta}\right\}^b, \tag{5}$$

where $\varphi = (a,b,c,k,\beta)^T$. Using (1), (2) and (4), the corresponding pdf of (5) is given by

$$f(x;\varphi) = \frac{a\,b\,ck\beta x^{c-1}}{(1+x^c)^{k+1}}\left[1-(1+x^c)^{-k}\right]^{a\beta-1}\left\{1-\left[1-(1+x^c)^{-k}\right]^{a\beta}\right\}^{b-1}, \tag{6}$$

where a,b,c,k and β are positive parameters. We denote a random variable X having pdf (6) by $X \sim$KEBXII(φ). The KEBXII distribution is very flexible and it has eleven sub-models when its parameters are carefully chosen. These special models are the Kumaraswamy Burr XII (KBXII), Kumaraswamy exponentiated Lomax (KEL), Kumaraswamy Lomax (KL), Kumaraswamy exponentiated log-logistic (KELL), Kumaraswamy log-logistic (KLL), exponentiated Burr XII (EBXII), exponentiated Lomax (EL), exponentiated log-logistic (ELL), BXII, Lomax (L) and log-logistic (LL) models. Table 1 lists these special models.

The survival function (sf), hrf and cumulative hazard rate function (chrf) of X are, respectively, given by

$$S(x;\varphi) = \left\{1-\left[1-(1+x^c)^{-k}\right]^{a\beta}\right\}^b,$$

$$h(x;\varphi) = \frac{a\,b\,ck\beta x^{c-1}}{(1+x^c)^{k+1}}\left[1-(1+x^c)^{-k}\right]^{a\beta-1}\left\{1-\left[1-(1+x^c)^{-k}\right]^{a\beta}\right\}^{-1},$$

and

$$H(x;\varphi) = -b\ln\left\{1-\left[1-(1+x^c)^{-k}\right]^{a\beta}\right\}.$$

Figure 1 display some plots of the KEBXII density for selected parameter values. Plots of the hrf of the KEBXII model for selected parameter values are given in Figure 2.

3. Theoretical Properties

In this section, we provide a mathematical treatment of the new KEBXII distribution including expansions for its pdf and cdf, moments, incomplete moments, quantile function, mean residual life and mean waiting time, Lorenz, Bonferroni and Zenga curves, order statistics and Rényi entropy.

Table 1: Sub-models of the KEBXII distribution.

a	b	c	k	β	Reduced Model	Author
–	–	–	–	1	KBXII	(Paranaíba et al., 2013)
–	–	1	–	–	KEL	(EL. Bata and Kareem, 2014)
–	–	1	–	1	KL	(Lemonte and Cordeiro, 2013)
–	–	–	1	–	KELL	New
–	–	–	1	1	KLL	(de Santana, Ortega, Cordeiro and Silva, 2012)
1	1	–	–	–	EBXII	
1	1	1	–	–	EL	
1	1	–	1	–	ELL	
1	1	–	–	1	BXII	
1	1	1	–	1	L	
1	1	–	1	1	LL	

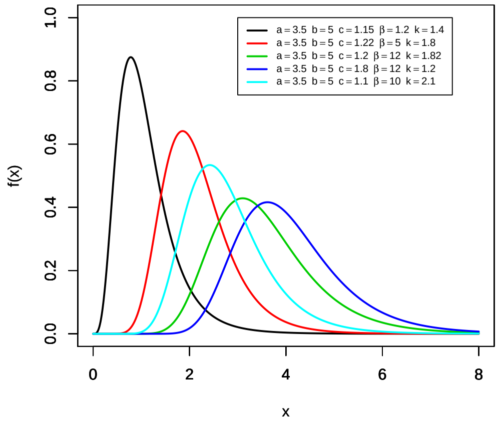

Figure 1: Plots of the KEBXII density function for some parameter values.

3.1. Mixture Representation

Expansions for the KEBXII density can be derived using the generalised binomial expansion given by

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{j!\Gamma(b-j)} z^j \text{ for } b>0 \text{ and } |z|<1. \tag{7}$$

Using expansion (7) in (6) and after some algebra, the pdf of X can be written as

$$f(x;\varphi) = abck\beta \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(b)}{i!\Gamma(b-i)} \frac{x^{c-1}}{(1+x^c)^{k+1}} \underbrace{\left[1-(1+x^c)^{-k}\right]^{a\beta(i+1)-1}}_{A}.$$

Applying (7) to the quantity denoted by A above, the last equation can be rewritten as

$$f(x;\varphi) = \sum_{j=0}^{\infty} q_j h(x;c,k(j+1)), \tag{8}$$

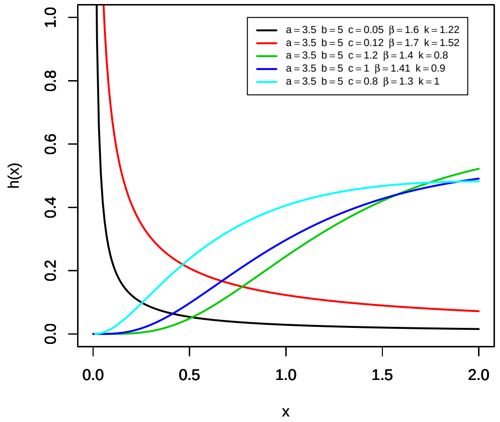

Figure 2: Plots of the KEBXII hrf for some parameter values.

where

$$q_j = ab\beta \sum_{i=0}^{\infty} \frac{(-1)^{i+j}}{(j+1)} \binom{b-1}{i} \binom{a\beta\,(i+1)-1}{j}$$

and $h(x;c,k(j+1))$ is the BXII density with parameters c and $k(j+1)$. Therefore, the KEBXII density can be expressed as a mixture of the BXII density and several of its structural properties can be obtained from (8) and the properties of the EBXII distribution.

Similarly, the cdf (5) of X can be expressed in the mixture form

$$F(x) = \sum_{j=0}^{\infty} q_j H(x;c,k(j+1)),$$

where $H(x;c,k(j+1))$ is the BXII cdf with parameters c and $k(j+1)$.

3.2. Moments

Let Y be a BXII distributed random variable, with parameters c and k. Then, the rth ordinary and incomplete moments of Y are given (for $r/c < k$) by

$$E(Y^r) = kB\left(1+\frac{r}{c}, k-\frac{r}{c}\right)$$

and

$$m_r(z) = kB\left(z^c; 1+\frac{r}{c}, k-\frac{r}{c}\right),$$

respectively, where $B(a,b) = \int_0^\infty t^{a-1}(1+t)^{-(a+b)}dt$ and $B(y;a,b) = \int_0^y t^{a-1}(1+t)^{-(a+b)}dt$ are the beta and the incomplete beta functions of the second type, respectively. It is noted, from the last equation, that $m_r(z) \to E(Y^r)$ when $z \to \infty$.

Then, using (8), the rth ordinary moment of X is given by

$$\mu_r' = E(X^r) = \sum_{j=0}^{\infty} q_j \int_0^{\infty} x^r\, h\left(x;c,k\left(j+1\right)\right) dx.$$

For $r/c < k$, we obtain

$$\mu_r' = \sum_{j=0}^{\infty} q_j\, k\left(j+1\right) B\left(1+\frac{r}{c}, k\left(j+1\right)-\frac{r}{c}\right). \tag{9}$$

Setting $r=1$ in (9), we have the mean of X. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships.

The nth central moment of X, say M_n, is given by

$$M_n = E(X-\mu)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\mu_1')^n\, \mu_{n-k}'.$$

3.3. Quantile Function

The qf of X is obtained by inverting (5) as

$$Q(u) = \left\{ \left\{ 1 - \left[1-(1-u)^{1/b}\right]^{1/a\beta} \right\}^{-1/k} - 1 \right\}^{1/c}, 0 < q < 1. \tag{10}$$

By setting $q = 0.5$ in (10) gives the median of X. Simulating the KEBXII random variable is straightforward. If U is a uniform variate on the unit interval $(0,1)$, then the random variable $X = Q(u)$ follows the distribution described in (5).

3.4. Incomplete Moments

The rth incomplete moment, say $m_r(t)$, of the KEBXII distribution is given by $m_r(t) = \int_0^t x^r f(x) dx$. We can write from (8)

$$m_r(t) = \sum_{j=0}^{\infty} q_j \int_0^t x^r\, h\left(x;c,k\left(j+1\right)\right).$$

Then, we obtain (for $r/c < k$)

$$m_r(t) = \sum_{j=0}^{\infty} q_j k\left(j+1\right) B\left(z^c; 1+\frac{r}{c}, k\left(j+1\right)-\frac{r}{c}\right).$$

Setting $r=1$ in the last equation, we obtain the first incomplete moment of X. The mean deviations about the mean and about the median of X can be expressed as $\delta_{\mu}(X) = \int_0^{\infty} |X-\mu_1'|\, f(x)dx = 2\mu_1' F(\mu_1') - 2m_1(\mu_1')$ and $\delta_M(X) = \int_0^{\infty} |X-M|\, f(x)dx = \mu_1' - 2m_1(M)$, respectively, where $\mu_1' = E(X)$ comes from (9), $F(\mu_1')$ is simply calculated from (5), $m_1(\mu_1')$ is the first incomplete moment and M is the median of X.

3.5. Mean Residual Life and Mean Waiting Time

The mean residual life (MRL) function or the life expectancy at a given time t, say $m(t)$, measures the expected remaining lifetime of an individual of age t and it given by

$$m(t)=\frac{1}{S(t)}\left[E(t)-\int_0^t tf(t)dt\right]-t.$$

Then, the MRL of X is given (for $k>1/c$) by

$$m(t)=\frac{\sum_{j=0}^{\infty}q_j k(j+1)B\left(1+\frac{1}{c},k(j+1)-\frac{1}{c}\right)-B\left(t^c;1+\frac{1}{c},k(j+1)-\frac{1}{c}\right)}{1-\sum_{j=0}^{\infty}q_j H(t;c,k(j+1))}-t.$$

The mean waiting time (MWT) or mean inactivity time represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in the interval $[0,t]$.The MWT of X, say $\overline{m}(t)$, is defined by

$$\overline{m}(t)=t-\frac{1}{F(t)}\int_0^t tf(t)dt.$$

For $k>1/c$, we obtain

$$\overline{m}(t)=t-\frac{\sum_{j=0}^{\infty}q_j k(j+1)B\left(t^c;1+\frac{1}{c},k(j+1)-\frac{1}{c}\right)}{\sum_{j=0}^{\infty}q_j H(t;c,k(j+1))}.$$

3.6. Lorenz, Bonferroni and Zenga Curves

Lorenz, Bonferroni and Zenga curves have been applied in many fields such as economics, reliability, demography, insurance and medicine. Further details can be found in Kleiber and Kotz (2003) and Zenga (2007). The Lorenz curve, say $L_F(x;\varphi)$, Bonferroni curve, say $B[F(x;\varphi)]$, and Zenga curve, say $A(x;\varphi)$, are given by

$$L_F(x;\varphi)=\frac{\sum_{j=0}^{\infty}q_j k(j+1)B\left(x^c;1+\frac{1}{c},k(j+1)-\frac{1}{c}\right)}{B\left(1+\frac{1}{c},k(j+1)-\frac{1}{c}\right)},$$

$$B[F(x;\varphi)]=\frac{\sum_{j=0}^{\infty}q_j k(j+1)B\left(x^c;1+\frac{1}{c},k(j+1)-\frac{1}{c}\right)}{\sum_{j=0}^{\infty}q_j H(x;c,k(j+1))B\left(1+\frac{1}{c},k(j+1)-\frac{1}{c}\right)}$$

and

$$A(x;\varphi)=1-\frac{\left[1-\sum_{j=0}^{\infty}q_j H(x;c,k(j+1))\right]B\left(x^c;1+\frac{1}{c},k(j+1)-\frac{1}{c}\right)}{\sum_{j=0}^{\infty}q_j H(x;c,k(j+1))\sum_{j=0}^{\infty}q_j k(j+1)P_j},$$

respectively, where $P_j=\left[B\left(1+\frac{1}{c},k(j+1)-\frac{1}{c}\right)-B\left(x^c;1+\frac{1}{c},k(j+1)-\frac{1}{c}\right)\right]$.

3.7. Order Statistics

If $X_1,\ldots,X_n$ is a random sample of size n from the KEBXII distribution and $X_{(1)},\ldots,X_{(n)}$ are the corresponding order statistics. Then, the pdf of the ith order statistic $X_{i:n}$, say $f_{i:n}(x)$, is given by

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-1}. \tag{11}$$

Using (5), (6) and (11) and, after some simplification, we can write

$$f_{i:n}(x) = \sum_{r=0}^{\infty} C_{i,n}^{(r)} h(x; c, k(r+1)), \tag{12}$$

where

$$C_{i,n}^{(r)} = \frac{n! ab\beta}{(i-1)!(n-i)!} \sum_{l,m=0}^{\infty} \frac{(-1)^{l+m+r}}{(r+1)} \binom{i-1}{l} \binom{a\beta(m+1)-1}{r} \binom{b(n+l-i+1)-1}{m}$$

and $h(x;c,k(r+1))$ is the BXII density function with parameters c and $k(r+1)$. Thus, the pdf of the KEBXII order statistics is a mixture of BXII densities. The pdf of the order statistic from a random sample of the BXII distribution comes by setting $a = b = \beta = 1$ in (12). Further, some of their mathematical properties can also be obtained from those of the BXII distribution.

For example, the sth moment of $X_{i:n}$ can be expressed as

$$E(X_{i:n}^s) = \sum_{r=0}^{\infty} C_{i,n}^{(r)} k(r+1)\, B\left(1+\frac{s}{c}, k(r+1)-\frac{s}{c}\right).$$

3.8. Rényi Entropy

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. The Rényi entropy is defined by

$$I_R(\delta) = \frac{1}{(1-\delta)} \log I(\delta),$$

where $I(\delta) = \int_{-\infty}^{\infty} f^{\delta}(x)dx,\ \delta > 0$ and $\delta \neq 1$.

Using (6), the binomial expansion and, after some simplification, we obtain

$$I(\delta) = (abck\beta)^{\delta} \sum_{i,j=0}^{\infty} \upsilon_{i,j} \int_0^{\infty} x^{\delta(c-1)} (1+x^c)^{-\delta(k+1)-kj} dx,$$

where $\upsilon_{i,j} = (-1)^{i+j} \binom{\delta(b-1)}{i} \binom{a\beta(\delta+i)-\delta}{j}$.

Using the transformation $y = x^c$ and, after some algebra, the last equation reduces to

$$I(\delta) = (abck\beta)^{\delta} \sum_{i,j=0}^{\infty} \upsilon_{i,j} B\left(\delta + \frac{(\delta-1)}{c}, k(\delta+j) - \frac{(\delta-1)}{c}\right).$$

Then, the Rényi entropy of the KEBXII distribution is given by

$$I_R(\delta) = \frac{1}{(1-\delta)} \log\left[(abck\beta)^{\delta} \sum_{i,j=0}^{\infty} \upsilon_{i,j} B\left(\delta + \frac{(\delta-1)}{c}, k(\delta+j) - \frac{(\delta-1)}{c}\right)\right].$$

4. Estimation

Here, we consider the estimation of the unknown parameters for this family from complete samples only by maximum likelihood. The MLEs of the parameters of the KEBXII $(x;\varphi)$ model is now discussed. Let $X_1,\ldots,X_n$ be a random sample of this distribution with unknown parameter vector $\varphi=(a,b,c,k,\beta)^{\intercal}$.

The log-likelihood function for φ, say $\ell=\ell(\varphi)$, is given by

$$\begin{aligned}\ell &= n(\ln a+\ln b+\ln c+\ln k+\ln\beta)+(b-1)\sum_{i=1}^{n}\ln\left(1-z_i^{a\beta}\right)\\ &\quad +(c-1)\sum_{i=1}^{n}\ln(p_i)^{1/c}-(k+1)\sum_{i=1}^{n}\ln(u_i)+(a\beta-1)\sum_{i=1}^{n}\ln(z_i),\end{aligned}$$

where $p_i=x_i^c$, $u_i=(1+p_i)$ and $z_i=\left(1-u_i^{-k}\right)$.

Then, the elements of the score vector components, $\mathbf{U}(\varphi)=\left(\frac{\partial\ell}{\partial a},\frac{\partial\ell}{\partial b},\frac{\partial\ell}{\partial c},\frac{\partial\ell}{\partial k},\frac{\partial\ell}{\partial\beta}\right)^{\intercal}$, are given by

$$\frac{\partial\ell}{\partial a}=\frac{n}{a}+\beta\sum_{i=1}^{n}\ln(z_i)-\beta(b-1)\sum_{i=1}^{n}\frac{z_i^{a\beta}\ln(z_i)}{1-z_i^{a\beta}},\quad \frac{\partial\ell}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\ln\left(1-z_i^{a\beta}\right),$$

$$\begin{aligned}\frac{\partial\ell}{\partial c} &= \frac{n}{c}-ak\beta(b-1)\sum_{i=1}^{n}\frac{z_i^{a\beta-1}u_i^{k-1}p_i\ln(p_i)^{1/c}}{1-z_i^{a\beta}}+\sum_{i=1}^{n}\ln(p_i)^{1/c}\\ &\quad -(k+1)\sum_{i=1}^{n}\frac{p_i\ln(p_i)^{1/c}}{u_i}+k(a\beta-1)\sum_{i=1}^{n}\frac{u_i^{k-1}p_i\ln(p_i)^{1/c}}{z_i},\end{aligned}$$

$$\frac{\partial\ell}{\partial k}=\frac{n}{k}-\sum_{i=1}^{n}\ln(u_i)-a\beta(b-1)\sum_{i=1}^{n}\frac{z_i^{a\beta-1}\ln(u_i)}{u_i^k\left(1-z_i^{a\beta}\right)}+(a\beta-1)\sum_{i=1}^{n}\frac{\ln(u_i)}{z_iu_i^k},$$

and

$$\frac{\partial\ell}{\partial\beta}=\frac{n}{\beta}+a\sum_{i=1}^{n}\ln(z_i)-a(b-1)\sum_{i=1}^{n}\frac{z_i^{a\beta}\ln(z_i)}{1-z_i^{a\beta}}.$$

We can obtain the estimates of the unknown parameters by setting the score vector to zero, $\mathbf{U}(\widehat{\varphi})=0$. By solving these equations simultaneously gives the MLEs $\widehat{a},\widehat{b},\widehat{c},\widehat{k}$ and $\widehat{\beta}$. They can not be solved analytically and statistical software can be used to solve them numerically by means of iterative techniques such as the Newton-Raphson algorithm.

5. Empirical Illustrations

In this section, we provide two applications to two real data sets to prove the importance and flexibility of the KEBXII distribution. The first data set represents Floyd river flood rates for the years 1935-1973 in Iowa, USA. Akinsete, Famoye and Lee (2008) and Alzaatreh, Famoye and Lee (2012) studied these data using the beta Pareto (BP) and gamma Pareto (GP) distributions, respectively. The second real data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients. These data were previously studied by Lee and Wang (2003), Lemonte and Cordeiro (2013) and Nofal, Afify, Yousof and Cordeiro (2017).

We compare the fit of the KEBXII distribution (and its sub-models namely: KEL, KELL, KBXII, EBXII and BXII distributions) with several other competitive models namely: the generalised inverse gamma (Mead, 2013), the beta exponentiated Burr XII (BEBXII) (Mead, 2014), the beta Fréchet (BF) (Nadarajah and Gupta, 2004; Barreto-Souza, Cordeiro and Simas, 2011), Kumaraswamy Fréchet (KF) (Mead and Abd-Eltawab, 2014), Kumaraswamy Pareto (KP) by Bourguignon et al. (2013), the Zografos-Balakrishnan log-logistic (ZBLL) by Zografos and Balakrishnan (2009) and the generalised gamma (GG) Stacy (1962) models with corresponding densities (for $x > 0$ except for the KP where $x > \beta$):

$$\begin{aligned}
\text{BEBXII:}\quad f(x;a,b,c,k,\beta) &= \tfrac{ck\beta}{B(a,b)} x^{c-1}\left(1+x^{c}\right)^{-(k+1)}\left[1-\left(1+x^{c}\right)^{-k}\right]^{a\beta-1} \\
&\quad \times\left\{1-\left[1-\left(1+x^{c}\right)^{-k}\right]^{\beta}\right\}^{b-1}; \\
\text{GIG:}\quad f(x;a,b,c,k,\beta) &= \tfrac{\beta c^{a\beta}}{\Gamma_b(a,k)} x^{-(a\beta+1)}\left[\left(\tfrac{c}{x}\right)^{\beta}+k\right]^{-\lambda}\exp\left[-\left(\tfrac{c}{x}\right)^{\beta}\right]; \\
\text{BF:}\quad f(x;a,b,c,\beta) &= \tfrac{\beta c^{\beta}}{B(a,b)} x^{-(\beta+1)}\exp\left[-a\left(\tfrac{c}{x}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\tfrac{c}{x}\right)^{\beta}\right]\right\}^{b-1}; \\
\text{KF:}\quad f(x;a,b,c,\beta) &= ab\beta c^{\beta} x^{-(\beta+1)}\exp\left[-a\left(\tfrac{c}{x}\right)^{\beta}\right]\left\{1-\exp\left[-a\left(\tfrac{c}{x}\right)^{\beta}\right]\right\}^{b-1}; \\
\text{KP:}\quad f(x;a,b,k,\beta) &= abk\beta^{k} x^{-(k+1)}\left[1-\left(\tfrac{\beta}{x}\right)^{k}\right]^{a-1}\left\{1-\left[1-\left(\tfrac{\beta}{x}\right)^{k}\right]^{a}\right\}^{b-1}; \\
\text{ZBLL:}\quad f(x;a,c,\beta) &= \tfrac{\beta c^{-\beta}}{\Gamma(a)} x^{\beta-1}\left[1+\left(\tfrac{x}{c}\right)^{\beta}\right]^{-2}\left\{\ln\left[1+\left(\tfrac{x}{c}\right)^{\beta}\right]\right\}^{a-1}; \\
\text{GG:}\quad f(x;a,c,\beta) &= \tfrac{\beta c^{-1}}{\Gamma(a)}\left(\tfrac{x}{c}\right)^{a\beta-1}\exp\left[-\left(\tfrac{x}{c}\right)^{\beta}\right];
\end{aligned}$$

where the parameters of the above densities are all positive real numbers, $\Gamma(a)$ is the gamma function and $\Gamma_b(a,k)$ is the generalised gamma function given by Kobayashi (1991) defined by

$$\Gamma_b(a,k) = \int_0^\infty y^{a-1}(k+y)^{-b}\exp(-y)\,dy.$$

In order to compare the models, we consider the Anderson-Darling (A^*) and Cramér-von Mises (W^*) statistics (full details can be found in Chen and Balakrishnan (1995)). In general, the model with minimum values for these statistics could be chosen as the best model to fit the data.

Tables 2 and 3 list the MLEs of the model parameters, their corresponding standard errors (given in parentheses) and the values of these statistics (A^* and W^*) for the fitted models to both data sets. The histogram of KEBXII estimated pdfs for the two data sets are displayed in Figures 3 and 4, respectively.

Tables 2 and 3 compare the KEBXII model with the KEL, KELL, KBXII, EBXII, BXII, BEBXII, GIG, BF, KF, KP, ZBLL and GG distributions. Note from Table 2 that the KEBXII model gives the lowest values for the A^* and W^* statistics (for the first data set) among all fitted models, whereas the KBXII has the lowest values for the A^* and W^* statistics (for the second data set) among all fitted models. So, the KEBXII and its sub-model (KBXII) distributions could be chosen as the best models. These results are obtained using the MATHCAD PROGRAM.

Table 2: MLEs, their corresponding standard errors and the statistics W^* and A^* for the first data set.

Model	Estimates					W^*	A^*
	$\widehat{a}$	$\widehat{b}$	$\widehat{c}$	$\widehat{\beta}$	$\widehat{k}$		
KEBXII	13.547292	4.246251	0.40436	13.812045	1.382952	0.021261	0.17069
	(139.824)	(7.897)	(2.13)	(142.557)	(9.082)		
KEL	21.86262	4.84026		5.73513	0.50895	0.02135	0.17233
	(239.625)	(5.635)		(62.86)	(0.244)		
KELL	12.13864	4.53497	0.53073	12.13627		0.021297	0.17132
	(145.266)	(4.753)	(0.217)	(145.238)			
KBXII	151.84771	4.52408	0.49323		1.08218	0.021267	0.17103
	(738.633)	(8.239)	(3.05)		(7.868)		
EBXII			3.07506	1347.14356	0.30482	0.050853	0.37001
			(19.16)	(906.746)	(1.899)		
EBXII			2.119		0.05765	0.09354	0.45421
			(63.722)		(1.745)		
BEBXII	49.54618	8.62295	0.27218	3.29085	1.32619	0.02157	0.17471
	(340.981)	(18.233)	(0.674)	(20.636)	(6.702)		
GIG	0.14461	0.01812	494.39805	3.66998	14.03693	0.11279	0.89762
	(0.0005)	(0.95205)	(3.006)	(0.502)	(9.490)		
BF	27.24953	10.1835	75.35864	0.30582		0.02151	0.1743
	(344.968)	(24.4296)	(694.38)	(0.38)			
KF	15.11457	560.73933	17.46945	0.15052		0.064948	0.47243
	(9.213)	(79.595)	(6.752)	(0.0312)			
KP	1	0.27318		318.1	1.5101	0.35694	2.56713
	(0.007)	(0.161)		-	(0.798)		
ZBLL	1.4568		1.5876	2087.7847		0.024976	0.17319
	(0.168)		(0.206)	(30.159)			
GG	0.28792		1.2289	10.44729		0.047635	0.35901
	(0.067)		(4.5)	(4.891)			

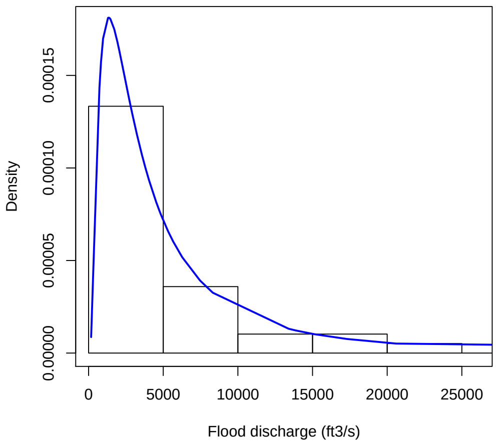

Figure 3: Plot of the estimated pdf of the KEBXII model for the first data set.

Table 3: MLEs, their corresponding standard errors and the statistics W^* and A^* for the second data set.

Model	Estimates					W^*	A^*
	$\widehat{a}$	$\widehat{b}$	$\widehat{c}$	$\widehat{\beta}$	$\widehat{k}$		
KEBXII	2.780009	67.636144	0.338304	3.082502	0.838911	0.04817	0.31966
	(44.51)	(104.728)	(0.385)	(49.353)	(1.723)		
KEL	1.59726	106.28888		1.7652	0.09534	0.105771	0.70421
	(5.914)	(125.189)		(6.536)	(0.054)		
KELL	3.25159	95.19093	0.28377	3.2516		0.048896	0.32061
	(19.642)	(182.458)	(0.104)	(19.642)			
KBXII	6.60303	118.58884	0.38837		0.55222	0.047695	0.31405
	(10.402)	(311.923)	(0.433)		(1.416)		
EBXII			0.65369	9.25526	1.90138	0.282415	1.86645
			(0.167)	(4.754)	(0.632)		
EBXII			2.33485		0.23375	0.693296	5.3726
			(0.354)		(0.04)		
BEBXII	22.18643	20.27685	0.22446	1.77993	1.30637	0.13369	0.90004
	(21.956)	(17.296)	(0.144)	(1.076)	(1.079)		
GIG	2.32748	0.00024	17.93139	0.54264	0.001	0.40999	2.61843
	(0.369)	(0.00019)	(7.385)	(0.042)	(0.0003)		
BF	6.12682	27.64441	92.4188	0.19843		0.16683	1.1078
	(6.135)	(9.1601)	(75.864)	(0.056)			
KF	5.15144	452.34073	24.10734	0.17058		0.05442	0.35912
	(18.399)	(83.5773)	(5.4230)	(0.0497)			
KP	4.95284	131.93483		0.08	0.10138	0.580129	3.66451
	(0.493)	(73.67)		-	(0.021)		
ZBLL	1.53969		1.54622	3.33888		0.15635	1.03729
	(0.093)		(0.111)	(0.037)			
GG	3.74791		0.5951	0.52009		0.050561	0.32851
	(2.635)		(1.425)	(0.195)			

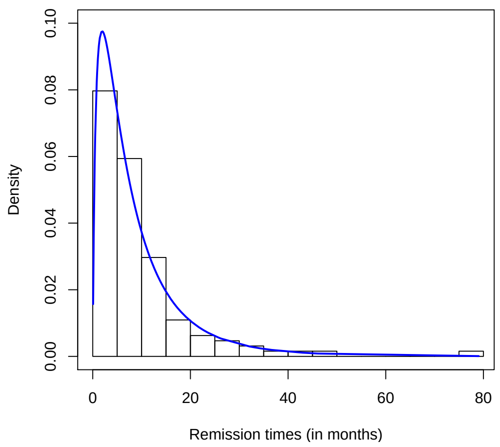

Figure 4: Plot of the estimated pdf of the KEBXII model for the second data set.

6. Conclusions

In this paper, we propose a new five-parameter distribution, called the Kumaraswamy exponentiated Burr XII (KEBXII) distribution, which extends the exponentiated Burr XII (EBXII) distribution and includes some well-known distributions as special cases. We provide some of its mathematical properties including the ordinary and incomplete moments, quantile and generating functions, Lorenz, Bonferroni and Zenga Curves, order statistics, mean residual life, mean waiting time and Rényi entropy. We discuss the maximum likelihood estimation of the model parameters. Two applications illustrate that the proposed model provides consistently better fit than other nested and non-nested models.

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References

AFIFY, A. Z., CORDEIRO, G. M., BUTT, N. S., ORTEGA, E. M. M., AND SUZUKI, A. K. (2016). A new lifetime model with variable shapes for the hazard rate. *Brazilian Journal of Probability and Statistics*. Forthcoming.

AKINSETE, A., FAMOYE, F., AND LEE, C. (2008). The beta Pareto distribution. *Statistics*, **42**, 547–563.

ALZAATREH, A., FAMOYE, F., AND LEE, C. (2012). Gamma Pareto distribution and its applications. *Journal of Modern Statistical Methods*, **11**, 78–95.

ANTONIO, E. G., DA SILVA, C. Q., AND CORDEIRO, G. (2014). Two extended Burr models: theory and practice. *Communications in Statistics-Theory and Methods*. Forthcoming.

BARRETO-SOUZA, W. M., CORDEIRO, G. M., AND SIMAS, A. B. (2011). Some results for beta Fréchet distribution. *Communications in Statistics-Theory and Methods*, **40**, 798–811.

BOURGUIGNON, M., SILVA, R., ZEA, L., AND CORDEIRO, G. (2013). The Kumaraswamy Pareto distribution. *Journal of Statistical Theory and Applications*, **12**, 129–144.

BURR, I. W. (1942). Cumulative frequency functions. *Annals of Mathematical Statistics*, **13**, 215–232.

CHEN, G. AND BALAKRISHNAN, N. (1995). A general purpose approximate goodness-of-fit test. *Journal of Quality Technology*, **27**, 154–161.

CORDEIRO, G. M. AND DE CASTRO, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, **81**, 883–898.

CORDEIRO, G. M., ORTEGA, E. M. M., AND NADARAJAH, S. (2010). The Kumaraswamy Weibull distribution with application to failure data. *Journal of the Franklin Institute*, **347**, 1399–1429.

CORDEIRO, G. M., PESCIM, R., AND ORTEGA, E. M. (2012). The Kumaraswamy generalized half-normal distribution for skewed positive data. *Journal of Data Science*, **10**, 195–224.

DE CASTRO, M., ORTEGA, E. M., AND CORDEIRO, G. M. (2011). The Kumaraswamy generalized gamma distribution with application in survival analysis. *Statistical Methodology*, **8**, 411–433.

DE SANTANA, T. V. F., ORTEGA, E. M., CORDEIRO, G. M., AND SILVA, G. O. (2012). The Kumaraswamy log-logistic distribution. *Journal of Statistical Theory and Applications*, **11**, 265–291.

EL. BATA, I. AND KAREEM, A. (2014). Statistical properties of Kumaraswamy exponentiated Lomax distribution. *Journal of Modern Mathematics and Statistics*, **8**, 1–7.

KLEIBER, C. AND KOTZ, S. (2003). *Statistical Size Distributions in Economics and Actuarial Sciences*. John Wiley and Sons: New York.

KOBAYASHI, K. (1991). On generalized gamma functions occurring in diffraction theory. *Journal Physical Society of Japan*, **60**, 1501–1512.

LEE, E. T. AND WANG, J. W. (2003). *Statistical Methods for Survival Data Analysis*. John Wiley and Sons: New York.

LEMONTE, A. J. AND CORDEIRO, G. M. (2013). An extended Lomax distribution. *Statistics: A Journal of Theoretical and Applied Statistics*, **47**, 800–816.

MEAD, M. E. (2013). Generalized inverse gamma distribution and its application in reliability. *Communications in Statistics-Theory and Methods*, **44**, 1426–1435.

MEAD, M. E. (2014). A new generalization of Burr XII distribution. *Journal of Statistics: Advances in Theory and Applications*, **12**, 53–71.

MEAD, M. E. AND ABD-ELTAWAB, A. R. (2014). A note on Kumaraswamy Fréchet distribution. *Australian Journal of Basic and Applied Sciences*, **8**, 294–300.

NADARAJAH, S. AND GUPTA, A. K. (2004). The beta Fréchet distribution. *Far East Journal of Theoretical Statistics*, **14**, 15–24.

NOFAL, Z. M., AFIFY, A. Z., YOUSOF, H. M., AND CORDEIRO, G. M. (2017). The generalized transmuted-G family of distributions. *Communications in Statistics-Theory and Methods*, **46**, 4119–4136.

PARANAÍBA, P. F., ORTEGA, E. M. M., CORDEIRO, G., AND DE PASCOA, M. (2013). The Kumaraswamy Burr XII distribution: theory and practice. *Journal of Statistical Computation and Simulation*, **83**, 2117–2143.

PARANAÍBA, P. F., ORTEGA, E. M. M., CORDEIRO, G. M., AND PESCIM, R. R. (2011). The beta Burr XII distribution with application to lifetime data. *Computational Statistics and Data Analysis*, **55**, 1118–1136.

SHAO, Q., WONG, H., XIA, J., AND IP, W. C. (2004). Models for extremes using the extended three-parameter Burr XII system with application to flood frequency analysis.
URL: `http://dx.doi.org/10.1623/hysj.49.4.685.54425`

STACY, E. W. (1962). A generalization of the gamma distribution. *Annals of Mathematical Statistics*, **33**, 1187–1192.

ZENGA, M. (2007). Inequality curve and inequality index based on the ratios between lower and upper arithmetic means. *Statistica and Applicazioni*, **1**, 3–27.

ZOGRAFOS, K. AND BALAKRISHNAN, N. (2009). On families of beta- and generalized gamma-generated distributions and associated inference. *Statistical Methodology*, **6**, 344–362.

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