EXTENDED VERSION OF GENERALISED LINDLEY DISTRIBUTION

M. R. Irshad
Department of Statistics, University College, Trivandrum -695 034, India.
e-mail: *irshadm24@gmail.com*

R. Maya
Department of Statistics, University College, Trivandrum -695 034, India.
e-mail: *mayasekharan@gmail.com*

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Abstract: In this paper, we consider an extended version of the generalised Lindley distribution ($EVGLD$). Some mathematical properties of the $EVGLD$ including moments, distribution of the order statistics, inequality measures, different entropy measures and vitality function are derived. The method of moments and method of maximum likelihood estimation are used to estimate the parameters. In addition to this, Fisher information matrix and asymptotic confidence interval are also included. Finally, two real life data sets are considered to illustrate the relevance of the new model and compared it with other forms of Lindley models.

1. Introduction

The Lindley distribution ($LD_1(\theta)$) was originally proposed by Lindley (1958) in the context of Bayesian Statistics, as a counter example of fiducial statistics with the probability density function (pdf)

$$f_1(x;\theta) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x};\; x > 0,\; \theta > 0,$$

where $f_1(x;\theta)$ is a mixture of exponential (θ) and gamma $(2,\theta)$ with mixing probabilities $\frac{\theta}{\theta+1}$ and $\frac{1}{\theta+1}$, respectively.

Although the Lindley distribution has drawn little attention in the statistical literature over the great popularity of the well known exponential distribution. Recently some researchers have proposed new classes of distributions based on modification of the one parameter Lindley distribution. Ghitany, Atieh and Nadarajah (2008) have studied various statistical properties of the Lindley distribution and described an application related to waiting time data. Mazucheli and Achcar (2011) applied the Lindley distribution to competing risk life time data. A discrete version of this distribution

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has been suggested by Deniz and Ojeda (2011) having its applications in count data related to insurance. Bakouch, Al-zahrani, Al-shomrani, Marchi and Louzada (2012) obtained an extended Lindley distribution and discussed its various properties and applications. Shanker and Mishra (2013a, 2013b)obtained generalised Lindley distributions and discussed their various properties and applications. Sah (2015) obtained a two-parameter Quasi-Lindley distribution and discussed their various properties. Pararai, Warahena-Liyanage and Oluyede (2015) proposed beta-exponentiated power Lindley (BEPL) distribution and studied some of its properties. Warahena-Liyanage and Pararai (2015) proposed a new class of lifetime distributions called the Lindley power series (LPS).

Shanker, Sharma and Shanker (2013) introduced a two-parameter Lindley distribution ($LD_2(\theta)$) with pdf

$$f_2(x;\alpha,\theta) = \frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x};\ x>0,\ \theta>0,\ \alpha>-\theta,$$

where $f_2(x;\alpha,\theta)$ is a mixture of exponential (θ) and gamma $(2,\theta)$ with mixing probabilities $\frac{\theta}{\theta+\alpha}$ and $\frac{\alpha}{\theta+\alpha}$ respectively.

Zakerzadeh and Dolati (2009) introduced a generalised Lindley distribution ($GLD_1(\alpha,\theta)$) with pdf

$$f_3(x;\alpha,\theta,\gamma) = \frac{\theta^2(\theta x)^{\alpha-1}(\alpha+\gamma x)}{(\gamma+\theta)\Gamma(\alpha+1)}e^{-\theta x};\ x>0,\ \alpha,\theta,\gamma>0,$$

where $f_3(x;\alpha,\theta,\gamma)$ is a mixture of gamma (α,θ) and gamma $(\alpha+1,\theta)$ with mixing probabilities $\frac{\theta}{\gamma+\theta}$ and $\frac{\gamma}{\gamma+\theta}$ respectively.

Abouammoh, Alshangiti and Ragab (2015) defined a new generalised Lindley distribution ($GLD_2(\alpha,\theta)$) with pdf

$$f_4(x;\alpha,\theta) = \frac{\theta^\alpha x^{\alpha-2}}{(\theta+1)\Gamma(\alpha)}(x+\alpha-1)e^{-\theta x}; x>0, \theta\geq 0, \alpha\geq 1,$$

where $f_4(x;\alpha,\theta)$ is a mixture of gamma (α,θ) and gamma $(\alpha-1,\theta)$ with mixing probabilities $\frac{1}{\theta+1}$ and $\frac{\theta}{\theta+1}$ respectively.

To increase the flexibility for modelling purposes it will be useful to consider further extensions of this distribution. Hence in this paper, our aim is to introduce an extended version of the generalised Lindley distribution called $EVGLD$, which offers a more flexible distribution for modelling life time data. The content of the paper is organised as follows. An extended version of generalised Lindley distribution is introduced in Section 2. The expressions for various reliability and statistical measures are derived in Section 3. Inequality measures such as Lorenz, Bonferroni and Zenga curves and different entropy measures such as Shannon's entropy, Havrda-Charvát-Tsallis entropy, Rényi entropy and Residual entropy are discussed in Section 4 and 5 respectively. In Section 6, we estimate the parameters by using method of moment estimation and method of maximum likelihood estimation. Fisher information and asymptotic confidence intervals are included in Section 7 and 8 respectively. Finally, in Section 9, two real life data sets are considered for comparing the performance of $EVGLD$ with the other Lindley forms of distributions.

2. Extended Version of Generalised Lindley Distribution

In this section, we define an extended version of the generalised Lindley distribution (*EVGLD*) and study some of its properties.

Let X be a non-negative random variable obtained from the mixture of two gamma distributions, namely gamma (α,θ) and gamma (β,θ) with mixing probabilities $p_1 = \frac{\theta k}{\eta^\delta+\theta k}$ and $p_2 = \frac{\eta^\delta}{\eta^\delta+\theta k}$ respectively. The corresponding pdf has the form

$$f(x;\theta,\alpha,\beta,k,\eta,\delta) = \frac{\theta^2}{\eta^\delta + \theta\, k}\left\{\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta^\delta(\theta x)^{\beta-1}}{\theta\Gamma(\beta)}\right\} e^{-\theta x}; x > 0, \tag{1}$$

with $\theta > 0$, $\alpha > 0$, $\beta > 0$, $\delta > 0$, $k \geq 0$, $\eta \geq 0$, subject to the constraint that k and η are not allowed to be simultaneously zero.

The cumulative distribution function (cdf) of *EVGLD* is given by

$$\begin{aligned} F(x;\theta,\alpha,\beta,k,\eta,\delta) &= \int_0^x f(t;\theta,\alpha,\beta,k,\eta,\delta)dt \\ &= \frac{1}{(\eta^\delta + \theta k)}\left\{\frac{\theta^2 k}{\Gamma(\alpha)}\int_0^x (\theta t)^{\alpha-1} e^{-\theta t} dt + \frac{\theta\eta^\delta}{\Gamma(\beta)}\int_0^x (\theta t)^{\beta-1} e^{-\theta t} dt\right\} \\ &= \frac{1}{(\eta^\delta + \theta k)}\Big\{\theta k \gamma_\alpha(\theta x) + \eta^\delta \gamma_\beta(\theta x)\Big\}, \end{aligned} \tag{2}$$

where

$$\gamma(a,b) = \int_0^b t^{a-1} e^{-t} dt$$

is the lower incomplete gamma function and

$$\gamma_a(b) = \frac{\gamma(a,b)}{\Gamma(a)}.$$

The survival function associated with (2) is obtained as

$$\overline{F}(x;\theta,\alpha,\beta,k,\eta,\delta) = 1 - \frac{1}{(\eta^\delta + \theta k)}\Big\{\theta k \gamma_\alpha(\theta x) + \eta^\delta \gamma_\beta(\theta x)\Big\}. \tag{3}$$

Remark 1 (1) If $\alpha = 1$, $\beta = 2$, $k = 1$, $\eta = 1$ and $\delta = 1$, then *EVGLD* becomes a one parameter Lindley distribution (LD_1) (see, Lindley, 1958).

(2) If $\alpha = 1$, $\beta = 2$, $k = 1$ and $\delta = 1$, then *EVGLD* becomes the two parameter Lindley distribution (LD_2) (see, Shanker et al., 2013).

(3) If $\beta = \alpha + 1$, $k = 1$ and $\delta = 1$, then *EVGLD* becomes the generalised Lindley distribution (GLD_1) (see, Zakerzadeh and Dolati, 2009).

(4) If $\alpha = \alpha - 1$, $\beta = \alpha$, $k = 1$, $\eta = 1$ and $\delta = 1$, then *EVGLD* becomes a new generalised Lindley distribution (GLD_2) (see, Abouammoh et al., 2015).

3. Statistical Properties

In this section, we look into some statistical properties of $EVGL$ random variables.

3.1. Reliability Measures

Let X be a continuous random variable with cdf $F(x)$ and pdf $f(x)$, then the hazard rate function, $h(x)$, cumulative hazard rate function, $R(x)$, reversed hazard rate function, $r(x)$, vitality function, $V(x)$, and mean residual life function, $m(x)$, are respectively given by $h(x)=\frac{f(x)}{\overline{F}(x)}$, $R(x)=-\ln\overline{F}(x)$, $r(x)=\frac{f(x)}{F(x)}$, $V(x)=\frac{1}{\overline{F}(x)}\int_x^\infty tf(t)dt$ and $m(x)=\frac{1}{\overline{F}(x)}\int_x^\infty tf(t)dt-x$.

Theorem 1 If X has the $EVGLD$ $(x;\theta,\alpha,\beta,k,\eta,\delta)$ with density function, cumulative distribution function and survival function given in (1), (2) and (3) respectively, then

a) Hazard rate function,

$$h(x)=\frac{\theta^2\left(\frac{k\left(\theta x\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\eta^\delta\left(\theta x\right)^{\beta-1}}{\theta\Gamma(\beta)}\right)e^{-\theta x}}{\eta^\delta+\theta k-\left\{\theta k\gamma_\alpha(\theta x)+\eta^\delta\gamma_\beta(\theta x)\right\}},$$

b) Cumulative hazard rate function,

$$R(x)=-\ln\left\{1-\frac{1}{(\eta^\delta+\theta k)}\left(\theta k\gamma_\alpha(\theta x)+\eta^\delta\gamma_\beta(\theta x)\right)\right\},$$

c) Reversed hazard rate function,

$$r(x)=\frac{\theta\left\{k\theta\left(\theta x\right)^{\alpha-1}\Gamma(\beta)+\eta^\delta\left(\theta x\right)^{\beta-1}\Gamma(\alpha)\right\}e^{-\theta x}}{\theta k\gamma(\alpha,\theta x)\Gamma(\beta)+\eta^\delta\gamma(\beta,\theta x)\Gamma(\alpha)},$$

d) Vitality function,

$$V(x)=\frac{\alpha\theta k\Gamma_{\alpha+1}(\theta x)+\beta\eta^\delta\Gamma_{\beta+1}(\theta x)}{\theta\left\{\eta^\delta+\theta k-\left(\theta k\gamma_\alpha(\theta x)+\eta^\delta\gamma_\beta(\theta x)\right)\right\}},$$

e) Mean residual life function,

$$m(x)=\frac{\alpha\theta k\Gamma_{\alpha+1}(\theta x)+\beta\eta^\delta\Gamma_{\beta+1}(\theta x)}{\theta\left\{\eta^\delta+\theta k-\left(\theta k\gamma_\alpha(\theta x)+\eta^\delta\gamma_\beta(\theta x)\right)\right\}}-x.$$

Proof. a) and b)
By using (1) and (3) in the equations $h(x)=\frac{f(x)}{\overline{F}(x)}$ and $R(x)=-\ln\overline{F}(x)$, the hazard rate and cumulative hazard rate functions are easily obtained.

The hazard rate function has different behaviour depending on its parameters. Figures 1–6 illustrate the hazard rate function of EVGLD for some selected parameters $\theta,\alpha,\beta,k,\eta$ and δ.

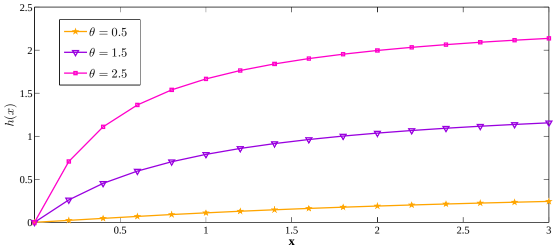

Figure 1: $\alpha=2,\beta=2.5,k=5,\eta=3,\delta=1.5.$

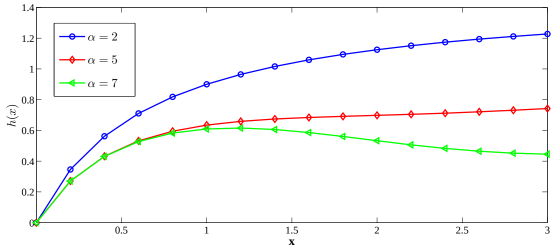

Figure 2: $\theta=1.5,\beta=2,k=2,\eta=5,\delta=1.5.$

c) By definition, we have

$$\begin{aligned} r(x) &= \frac{f(x)}{F(x)} \\ &= \frac{\theta^2\{k\theta(\theta x)^{\alpha-1}\Gamma(\beta)+\eta^\delta(\theta x)^{\beta-1}\Gamma(\alpha)\}e^{-\theta x}}{\theta\Gamma(\alpha)\Gamma(\beta)\{\theta k\gamma_\alpha(\theta x)+\eta^\delta\gamma_\beta(\theta x)\}} \\ &= \frac{\theta\{k\theta(\theta x)^{\alpha-1}\Gamma(\beta)+\eta^\delta(\theta x)^{\beta-1}\Gamma(\alpha)\}e^{-\theta x}}{\theta k\gamma(\alpha,\theta x)\Gamma(\beta)+\eta^\delta\gamma(\beta,\theta x)\Gamma(\alpha)}. \end{aligned}$$

d) We have

$$V(x)=\frac{1}{\overline{F}(x)}\int_x^\infty tf(t)dt. \tag{4}$$

Now,

$$\begin{aligned} \int_x^\infty tf(t)dt &= \frac{1}{\theta(\eta^\delta+\theta k)}\left\{\frac{\theta k}{\Gamma(\alpha)}\Gamma(\alpha+1,\theta x)+\frac{\eta^\delta}{\Gamma(\beta)}\Gamma(\beta+1,\theta x)\right\} \\ &= \frac{1}{\theta(\eta^\delta+\theta k)}\left\{\alpha\theta k\Gamma_{\alpha+1}(\theta x)+\beta\eta^\delta\Gamma_{\beta+1}(\theta x)\right\}, \end{aligned} \tag{5}$$

where

$$\Gamma(a,b)=\int_b^\infty t^{a-1}e^{-t}dt,$$

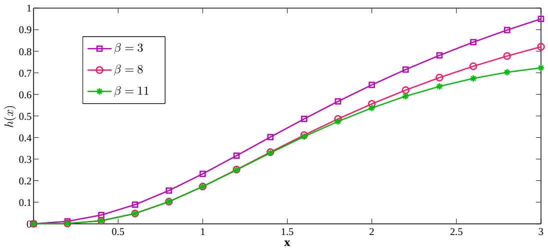

Figure 3: $\theta = 2, \alpha = 5, k = 7, \eta = 2, \delta = 0.5$.

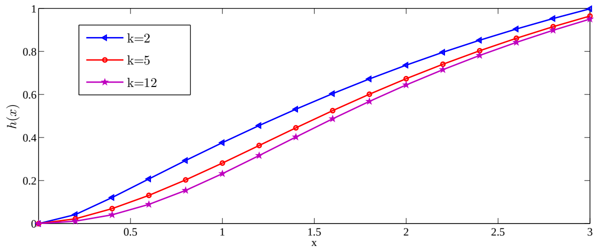

Figure 4: $\theta = 2, \alpha = 5, \beta = 3, \eta = 6, \delta = 0.5$.

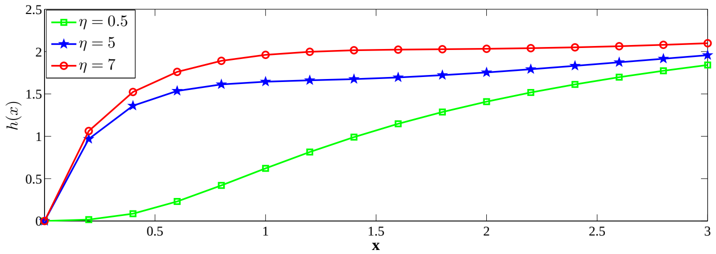

Figure 5: $\theta = 3, \alpha = 5, \beta = 2, k = 6, \delta = 3$.

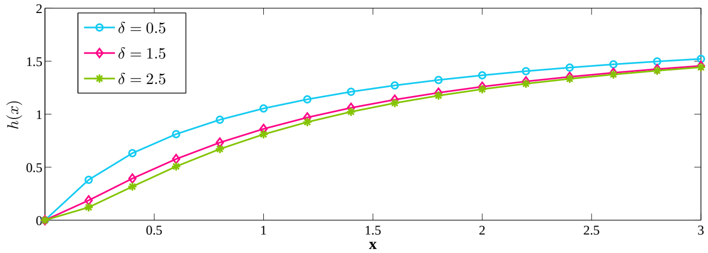

Figure 6: $\theta = 2, \alpha = 2, \beta = 3, k = 2, \eta = 7$.

is the upper incomplete gamma function and

$$\Gamma_a(b) = \frac{\Gamma(a,b)}{\Gamma(a)}.$$

Substituting the values of (3) and (5) in (4), we obtain

$$V(x) = \frac{\alpha\theta k\Gamma_{\alpha+1}(\theta x) + \beta\eta^{\delta}\Gamma_{\beta+1}(\theta x)}{\theta\left\{\eta^{\delta} + \theta k - \left(\theta k\gamma_{\alpha}(\theta x) + \eta^{\delta}\gamma_{\beta}(\theta x)\right)\right\}}.$$

e) We have

$$\begin{aligned} m(x) &= \frac{1}{\overline{F}(x)}\int_x^{\infty} t f(t)dt - x \\ &= V(x) - x. \end{aligned}$$

Substituting the value of $V(x)$ into the above equation, the mean residual function for $EVGLD$ is obtained. ■

3.2. Moments and Associated Measures

Theorem 2 If X has the $EVGLD$ $(x;\theta,\alpha,\beta,k,\eta,\delta)$ with density function given in (1), then

a) The r^{th} raw moment $\mu_r^{'}$ about the origin is given by

$$\mu_r^{'} = \frac{1}{\eta^{\delta} + \theta k}\left\{\frac{k\Gamma(\alpha+r)}{\theta^{r-1}\Gamma(\alpha)} + \frac{\eta^{\delta}\Gamma(\beta+r)}{\theta^{r}\Gamma(\beta)}\right\}. \tag{6}$$

b) The moments of the $EVGL$ random variable can be calculated recursively through the relationship

$$\begin{aligned} \mu_{r+1}^{'} = \frac{(\alpha+\beta+2r)}{\theta}\mu_r^{'} - \frac{1}{\eta^{\delta}+\theta k}\Bigg\{&\frac{k(\beta+r)\Gamma(\alpha+r)}{\theta^{r}\Gamma(\alpha)} + \\ &\frac{\eta^{\delta}(\alpha+r)\Gamma(\beta+r)}{\theta^{r+1}\Gamma(\beta)}\Bigg\}. \end{aligned} \tag{7}$$

c) The moment generating function (mgf) of the $EVGLD$ $(x;\theta,\alpha,\beta,k,\eta,\delta)$ is given by

$$M_X(t) = \frac{1}{\eta^{\delta}+\theta k}\left\{\sum_{j=0}^{\infty}\frac{(-1)^j}{\theta^j}\left\{\theta k\binom{-\alpha}{j} + \eta^{\delta}\binom{-\beta}{j}\right\}t^j\right\}.$$

Proof. a) By the definition, we have

$$\begin{aligned} \mu_r^{'} &= E(X^r) \\ &= \int_0^{\infty} x^r f(x)dx \\ &= \frac{1}{\eta^{\delta}+\theta k}\left\{\frac{k\theta^{\alpha+1}}{\Gamma(\alpha)}\int_0^{\infty} x^{r+\alpha-1}e^{-\theta x}dx + \frac{\eta^{\delta}\theta^{\beta}}{\Gamma(\beta)}\int_0^{\infty} x^{r+\beta-1}e^{-\theta x}dx\right\} \\ &= \frac{1}{\eta^{\delta}+\theta k}\left\{\frac{k\theta^{1-r}\Gamma(r+\alpha)}{\Gamma(\alpha)} + \frac{\eta^{\delta}\theta^{-r}\Gamma(r+\beta)}{\Gamma(\beta)}\right\}. \end{aligned}$$

By rearranging the above equation, we obtain (6).

Result 1

The mean, variance and coefficient of variation of the EVGL random variable are respectively given by

$$Mean,\ \mu_1^{'} = \frac{\alpha\theta k + \eta^{\delta}\beta}{\theta(\eta^{\delta}+\theta k)},$$

$$Variance,\ \mu_2 = \frac{1}{\theta^2(\eta^{\delta}+\theta k)^2}\Big\{\eta^{\delta}\theta k\Big((\alpha-\beta)^2+(\alpha+\beta)\Big)+\beta\eta^{2\delta}+\alpha\theta^2k^2\Big\}$$

and

$$Coefficient\ of\ variation,\ \zeta = \frac{\sqrt{\eta^{\delta}\theta k\Big((\alpha-\beta)^2+(\alpha+\beta)\Big)+\beta\eta^{2\delta}+\alpha\theta^2k^2}}{\alpha\theta k+\eta^{\delta}\beta}\times 100.$$

b) From (6), we have

$$(\eta^{\delta}+\theta k)\mu_r^{'} = \frac{k\Gamma(\alpha+r)}{\theta^{r-1}\Gamma(\alpha)}+\frac{\eta^{\delta}\Gamma(\beta+r)}{\theta^{r}\Gamma(\beta)}.$$

Put $r = r+1$ in the above equation, we obtain

$$(\eta^{\delta}+\theta k)\mu_{r+1}^{'} = \frac{k\Gamma(\alpha+r+1)}{\theta^{r}\Gamma(\alpha)}+\frac{\eta^{\delta}\Gamma(\beta+r+1)}{\theta^{r+1}\Gamma(\beta)}.$$

That is,

$$\begin{aligned}\theta(\eta^{\delta}+\theta k)\mu_{r+1}^{'} &= (\alpha+r)\left\{(\eta^{\delta}+\theta k)\mu_r^{'}-\frac{\eta^{\delta}\Gamma(\beta+r)}{\theta^{r}\Gamma(\beta)}\right\}+\\ &\quad(\beta+r)\left\{(\eta^{\delta}+\theta k)\mu_r^{'}-\frac{k\Gamma(\alpha+r)}{\theta^{r-1}\Gamma(\alpha)}\right\}\\ &= \mu_r^{'}(\eta^{\delta}+\theta k)(\alpha+\beta+2r)-\frac{\eta^{\delta}(\alpha+r)\Gamma(\beta+r)}{\theta^{r}\Gamma(\beta)}-\\ &\quad\frac{k(\beta+r)\Gamma(\alpha+r)}{\theta^{r-1}\Gamma(\alpha)}.\end{aligned}$$

By rearranging the above equation, we obtain (7).

c) By using the definition of the moment generating function, we have

$$\begin{aligned}M_X(t) = E(e^{tX}) &= \int_0^{\infty} e^{tx}f(x;\theta,\alpha,\beta,k,\eta,\delta)dx\\ &= \frac{1}{\eta^{\delta}+\theta k}\left\{\frac{k\theta^{\alpha+1}}{\Gamma(\alpha)}\int_0^{\infty}x^{\alpha-1}e^{-(\theta-t)x}dx+\frac{\eta^{\delta}\theta^{\beta}}{\Gamma(\beta)}\int_0^{\infty}x^{\beta-1}e^{-(\theta-t)x}dx\right\}\\ &= \frac{1}{\eta^{\delta}+\theta k}\left\{\frac{k\theta^{\alpha+1}}{(\theta-t)^{\alpha}}+\frac{\theta^{\beta}\eta^{\delta}}{(\theta-t)^{\beta}}\right\}\\ &= \frac{1}{\eta^{\delta}+\theta k}\left\{k\theta\left(1-\frac{t}{\theta}\right)^{-\alpha}+\eta^{\delta}\left(1-\frac{t}{\theta}\right)^{-\beta}\right\}\\ &= \frac{1}{\eta^{\delta}+\theta k}\left\{\sum_{j=0}^{\infty}\frac{(-1)^j}{\theta^j}\left[\theta k\binom{-\alpha}{j}+\eta^{\delta}\binom{-\beta}{j}\right]t^j\right\}.\end{aligned}$$

Result 2
The characteristic function of EVGLD $(x;\theta,\alpha,\beta,k,\eta,\delta)$ *becomes* $\phi_X(t) = M_X(it)$*, where* $i=\sqrt{-1}$ *is the unit imaginary number.* ■

3.3. Distribution of Order Statistics

Let $X_1, X_2, \ldots, X_n$ denote n independent random variables from a distribution with distribution function $F(x)$ and probability density function $f(x)$, then the pdf of the r^{th} order statistic $X_{r:n}$ is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} f(x) \Big(F(x)\Big)^{r-1} \Big(1-F(x)\Big)^{n-r}, \tag{8}$$

$r = 1, 2, \ldots, n$.
The pdf of the r^{th} order statistic $X_{r:n}$ for the $EVGL$ random variable is given by

$$\begin{aligned} f_{r:n}(x,\theta,\alpha,\beta,k,\eta,\delta) = {} & \frac{n!}{(r-1)!(n-r)!} \frac{\theta^2}{(\eta^\delta+\theta k)^n} \left(\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta^\delta(\theta x)^{\beta-1}}{\theta\Gamma(\beta)} \right) e^{-\theta x} \\ & \times \Big\{ \theta k \Big(1-\gamma_\alpha(\theta x)\Big) + \eta^\delta \Big(1-\gamma_\beta(\theta x)\Big) \Big\}^{n-r} \\ & \times \Big\{ \theta k \gamma_\alpha(\theta x) + \eta^\delta \gamma_\beta(\theta x) \Big\}^{r-1}. \end{aligned}$$

Setting $r = 1$ and $r = n$ in (8), the pdf of the 1^{st} order statistic $X_{1:n}$ and n^{th} order statistic $X_{n:n}$ are then respectively given by

$$f_{1:n}(x) = n\Big(1-F(x)\Big)^{n-1} f(x) \tag{9}$$

and

$$f_{n:n}(x) = n\Big(F(x)\Big)^{n-1} f(x). \tag{10}$$

Substituting (1) and (2) in (9) and (10), we obtain the pdf of the 1^{st} order statistic $X_{1:n}$ and the n^{th} order statistic $X_{n:n}$ for the $EVGL$ random variable, which are respectively given by

$$\begin{aligned} f_{1:n}(x,\theta,\alpha,\beta,k,\eta,\delta) = {} & \frac{n\theta^2}{(\eta^\delta+\theta k)^n} \left(\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta^\delta(\theta x)^{\beta-1}}{\theta\Gamma(\beta)} \right) e^{-\theta x} \\ & \times \Big\{ \theta k \Big(1-\gamma_\alpha(\theta x)\Big) + \eta^\delta \Big(1-\gamma_\beta(\theta x)\Big) \Big\}^{n-1} \end{aligned}$$

and

$$\begin{aligned} f_{n:n}(x,\theta,\alpha,\beta,k,\eta,\delta) = {} & \frac{n\theta^2}{(\eta^\delta+\theta k)^n} \left(\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)} + \frac{\eta^\delta(\theta x)^{\beta-1}}{\theta\Gamma(\beta)} \right) e^{-\theta x} \\ & \times \Big\{ \theta k \gamma_\alpha(\theta x) + \eta^\delta \gamma_\beta(\theta x) \Big\}^{n-1}. \end{aligned}$$

4. Inequality Measures

The expressions for the inequality measures such as Lorenz, Bonferroni and Zenga curves of $EVGL$ random variable are derived in this section.

Bonferroni and Lorenz curves (see, Bonferroni, 1930) have been used in economics to study income and poverty. These curves have many applications in other fields such as demography, reliability, insurance and medicine and engineering. The Lorenz and Bonferroni curves are respectively given as $L_F(x) = (\int_0^x tf(t)dt)/E(X)$ and $B(F(x)) = (\int_0^x tf(t)dt)/(F(X)E(X))$.

Also, the Zenga curve introduced by Zenga (2007) is another widely used inequality measure and is given as $A(x) = 1 - \mu^-(x)/\mu^+(x)$, where $\mu^-(x) = (\int_0^x tf(t)dt)/F(X)$ and $\mu^+(x) = (\int_x^\infty tf(t)dt)/\overline{F}(X)$.

Theorem 3 If X has the $EVGLD\ (x;\theta,\alpha,\beta,k,\eta,\delta)$ with density function, cumulative distribution function and survival function given in (1), (2) and (3) respectively, then

a) Lorenz curve,

$$L_F(x) = \frac{\alpha\theta k\gamma_{\alpha+1}(\theta x) + \beta\eta^\delta\gamma_{\beta+1}(\theta x)}{\alpha\theta k + \beta\eta^\delta}.$$

b) Bonferroni curve,

$$B(F(x)) = \frac{(\eta^\delta + \theta k)(\alpha\theta k\gamma_{\alpha+1}(\theta x) + \beta\eta^\delta\gamma_{\beta+1}(\theta x))}{(\alpha\theta k + \beta\eta^\delta)(\theta k\gamma_\alpha(\theta x) + \eta^\delta\gamma_\beta(\theta x))}.$$

c) Zenga curve,

$$A(x) = 1 - \left\{ \frac{\Big(\alpha\theta k\gamma_{\alpha+1}(\theta x) + \beta\eta^\delta\gamma_{\beta+1}(\theta x)\Big)}{\Big(\alpha\theta k\Gamma_{\alpha+1}(\theta x) + \beta\eta^\delta\Gamma_{\beta+1}(\theta x)\Big)} \right. \\ \left. \times \frac{\Big(\theta k\big(1-\gamma_\alpha(\theta x)\big) + \eta^\delta\big(1-\gamma_\beta(\theta x)\big)\Big)}{\Big(\theta k\gamma_\alpha(\theta x) + \eta^\delta\gamma_\beta(\theta x)\Big)} \right\}.$$

Proof. a) We have

$$L_F(x) = \frac{\int_0^x tf(t)dt}{E(X)}.$$

Now

$$\begin{aligned}\int_0^x tf(t)dt &= \frac{1}{\theta(\eta^\delta + \theta k)} \left\{ \frac{\theta k}{\Gamma(\alpha)}\gamma(\alpha+1,\theta x) + \frac{\eta^\delta}{\Gamma(\beta)}\gamma(\beta+1,\theta x) \right\} \\ &= \frac{1}{\theta(\eta^\delta + \theta k)} \Big\{ \alpha\theta k\gamma_{\alpha+1}(\theta x) + \beta\eta^\delta\gamma_{\beta+1}(\theta x) \Big\}.\end{aligned}$$

Therefore,

$$L_F(x) = \frac{\frac{1}{\theta(\eta^\delta+\theta k)}\left\{\alpha\theta k\gamma_{\alpha+1}(\theta x)+\beta\eta^\delta\gamma_{\beta+1}(\theta x)\right\}}{\frac{\alpha\theta k+\eta^\delta\beta}{\theta(\eta^\delta+\theta k)}}$$
$$= \frac{\alpha\theta k\gamma_{\alpha+1}(\theta x)+\beta\eta^\delta\gamma_{\beta+1}(\theta x)}{\alpha\theta k+\beta\eta^\delta}.$$

b) We have

$$B(F(x)) = \frac{\int_0^x tf(t)dt}{F(X)E(X)}$$
$$= \left\{\frac{\alpha\theta k\gamma_{\alpha+1}(\theta x)+\beta\eta^\delta\gamma_{\beta+1}(\theta x)}{\alpha\theta k+\beta\eta^\delta}\right\}\left\{\frac{\eta^\delta+\theta k}{\theta k\gamma_\alpha(\theta x)+\eta^\delta\gamma_\beta(\theta x)}\right\}$$
$$= \frac{(\eta^\delta+\theta k)(\alpha\theta k\gamma_{\alpha+1}(\theta x)+\beta\eta^\delta\gamma_{\beta+1}(\theta x))}{(\alpha\theta k+\beta\eta^\delta)(\theta k\gamma_\alpha(\theta x)+\eta^\delta\gamma_\beta(\theta x))}.$$

c) We have

$$A(x) = 1-\frac{\mu^-(x)}{\mu^+(x)}.$$

Now,

$$\mu^-(x) = \frac{\int_0^x tf(t)dt}{F(X)}$$
$$= \frac{\frac{1}{\theta(\eta^\delta+\theta k)}\left\{\alpha\theta k\gamma_{\alpha+1}(\theta x)+\beta\eta^\delta\gamma_{\beta+1}(\theta x)\right\}}{\frac{1}{(\eta^\delta+\theta k)}\left\{\theta k\gamma_\alpha(\theta x)+\eta^\delta\gamma_\beta(\theta x)\right\}}$$
$$= \frac{\alpha\theta k\gamma_{\alpha+1}(\theta x)+\beta\eta^\delta\gamma_{\beta+1}(\theta x)}{\theta(\theta k\gamma_\alpha(\theta x)+\eta^\delta\gamma_\beta(\theta x))}.$$

$$\mu^+(x) = \frac{\int_x^\infty tf(t)dt}{\overline{F}(X)}$$
$$= V(x).$$

Substituting $V(x)$ given in Theorem 1., we obtain

$$\mu^+(x) = \frac{\alpha\theta k\Gamma_{\alpha+1}(\theta x)+\beta\eta^\delta\Gamma_{\beta+1}(\theta x)}{\theta\left\{\theta k(1-\gamma_\alpha(\theta x))+\eta^\delta(1-\gamma_\beta(\theta x))\right\}}.$$

Therefore,

$$A(x) = 1-\left\{\frac{\alpha\theta k\gamma_{\alpha+1}(\theta x)+\beta\eta^\delta\gamma_{\beta+1}(\theta x)}{\alpha\theta k\Gamma_{\alpha+1}(\theta x)+\beta\eta^\delta\Gamma_{\beta+1}(\theta x)}\right.$$
$$\left.\times\frac{\theta k(1-\gamma_\alpha(\theta x))+\eta^\delta(1-\gamma_\beta(\theta x))}{\theta k(\gamma_\alpha(\theta x))+\eta^\delta(\gamma_\beta(\theta x))}\right\}.$$

Thus the theorem is proved. ■

5. Uncertainty Measures

In this section, we focus the attention on various entropy measures.

The concept of entropy was introduced and extensively studied by Shannon (1948). Let X be a non-negative random variable admitting an absolutely continuous cdf $F(x)$ and with pdf $f(x)$. Then the Shannon's entropy associated with X is defined as $H(X) = -\int_0^\infty f(x)\ \ln f(x)\ dx$. It gives the expected uncertainty contained in $f(x)$ about the predictability of an outcome of X. The Shannon's entropy finds immense applications in several branches of learning. In Communication theory, an aspect of interest is the flow of information in some network where information is carried from a transmitter to receiver. Theil (1967) discussed the applications of information theory to problems in Economics such as the measurement of income inequality, industrial concentration, concentration in international trade and the fit of allocation models. Ecologists measure the diversity of a species in biological populations using entropy (see, Pielou, 1967). In Cryptography, Shannon's entropy is used as a cryptographic measure for the key generator module, which forms the part of the security of the cipher system (see, Simion, 2000).

Several generalizations of Shannon's entropy have been put forward by researchers. A generalization which has received much attention subsequently is due to Rényi (1961). The Rényi's entropy of order ν is defined as $H^\nu(X) = \frac{1}{1-\nu} \ln \int_0^\infty f^\nu(x)\ dx,$ for $\nu > 0, \nu \neq 1$. Another important generalization of Shannon's entropy is the Havrda-Charvát-Tsallis (HCT) entropy. It was introduced by Havrada and Charvát (1967) and further developed by Tsallis (1988) and Gell-Mann and Tsallis (2004) and is given by $H^\rho(X) = \frac{1}{\rho-1}\left(1 - \int_0^\infty f^\rho(x)\ dx\right)$, for $\rho > 0, \rho \neq 1$.

If we think of X as the life-time of a new unit, then $H(X)$ can be viewed as a useful tool for measuring the associated uncertainty. However, if a unit is known to have survived up to age x, then $H(X)$ is no longer useful for measuring the uncertainty about remaining life-time of the unit. In this scenario, Ebrahimi and Pellerey (1995) followed by Ebrahimi (1996) have proposed the concept of residual entropy and is defined as $H(f;x) = \ln \overline{F}(x) - \frac{1}{\overline{F}(x)} \int_x^\infty f(t)\ \ln f(t)\ dt$.

Theorem 4 If X has the $EVGLD\ (x;\theta,\alpha,\beta,k,\eta,\delta)$ with density function, cumulative distribution function and survival function given in (1), (2) and (3) respectively, then

a) Shannon's entropy,

$$\begin{aligned} H(X) = -\frac{\theta^2}{\eta^\delta + \theta k} \Bigg\{ \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \left(\frac{\eta^\delta \Gamma(\alpha)}{k\theta\Gamma(\beta)}\right)^j \Bigg(\frac{k}{\theta} \frac{\Gamma\Big(\alpha + j(\beta-\alpha)\Big)}{\Gamma(\alpha)} + \\ \frac{\eta^\delta}{\theta^2} \frac{\Gamma\Big(\beta + j(\beta-\alpha)\Big)}{\Gamma(\beta)} \Bigg) + \frac{k(\alpha-1)}{\theta}\psi(\alpha) + \\ \frac{(\alpha-1)}{\theta^2}\eta^\delta\psi(\beta) - \frac{k\alpha}{\theta} - \frac{\eta^\delta\beta}{\theta^2} \Bigg\} - \ln\left(\frac{k\theta^2}{(\eta^\delta+\theta k)\Gamma(\alpha)}\right), \end{aligned} \tag{11}$$

where $\psi(.)$ is the digamma function and is given by

$$\psi(a) = \frac{d}{da} \ln\Big(\Gamma(a)\Big) = \frac{\Gamma^{'}(a)}{\Gamma(a)}$$

and

$$\Gamma^{'}(a) = \int_0^\infty t^{a-1}\ \ln(t)\ e^{-t} dt$$

is the first derivative of the gamma function.

b) Rényi's entropy of order ν,

$$H^{\nu}(X) = \frac{1}{1-\nu}\left\{\ln\left(\frac{k\theta^{\frac{2\nu-1}{\nu}}}{\nu^{\frac{(\alpha-1)\nu+1}{\nu}}(\eta^\delta+\theta k)\Gamma(\alpha)}\right)^{\nu} + \ln\left(\sum_{j=0}^{\nu}\binom{\nu}{j}\left(\frac{\eta^\delta\Gamma(\alpha)}{\theta k\Gamma(\beta)}\right)^j \frac{\Gamma\Big(\nu(\alpha-1)+j(\beta-\alpha)+1\Big)}{\nu^{j(\beta-\alpha)}}\right)\right\},$$

for $\nu > 0, \nu \neq 1$.

c) Havrda-Charvát-Tsallis entropy of order ρ,

$$H^{\rho}(X) = \frac{1}{\rho-1}\left\{1-\left(\frac{k\theta^{\frac{2\rho-1}{\rho}}}{\rho^{\frac{(\alpha-1)\rho+1}{\rho}}(\eta^\delta+\theta k)\Gamma(\alpha)}\right)^{\rho} \times\sum_{j=0}^{\rho}\binom{\rho}{j}\left(\frac{\eta^\delta\Gamma(\alpha)}{\theta k\Gamma(\beta)}\right)^j \frac{\Gamma\Big(\rho(\alpha-1)+j(\beta-\alpha)+1\Big)}{\rho^{j(\beta-\alpha)}}\right\},$$

for $\rho > 0$, $\rho \neq 1$.

d) Residual entropy,

$$\begin{aligned} H(f;x) = {} & \ln\left(\overline{F}(x)\right) - \ln\left(\frac{\theta^2 k}{\Gamma(\alpha)(\eta^\delta+\theta k)}\right) - \frac{\theta^2}{\overline{F}(x)(\eta^\delta+\theta k)}\times \\ & \left\{\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\left(\frac{\eta^\delta\Gamma(\alpha)}{k\theta\Gamma(\beta)}\right)^j\left(\frac{k}{\theta}\frac{\Gamma\Big(\alpha+j(\beta-\alpha),\theta x\Big)}{\Gamma(\alpha)}+\right.\right. \\ & \left.\frac{\eta^\delta}{\theta^2}\frac{\Gamma\Big(\beta+j(\beta-\alpha),\theta x\Big)}{\Gamma(\beta)}\right)+\frac{k(\alpha-1)}{\theta}\frac{\Gamma^{'}\Big(\alpha,\theta x\Big)}{\Gamma(\alpha)}+ \\ & \frac{(\alpha-1)\eta^\delta}{\theta^2}\frac{\Gamma^{'}\Big(\beta,\theta x\Big)}{\Gamma(\beta)}-\frac{k}{\theta}\frac{\Gamma\Big(\alpha+1,\theta x\Big)}{\Gamma(\alpha)}- \\ & \left.\frac{\eta^\delta}{\theta^2}\frac{\Gamma\Big(\beta+1,\theta x\Big)}{\Gamma(\beta)}\right\}, \end{aligned}$$

where $\overline{F}(x)$ is given in (3) and

$$\Gamma^{'}(a,b) = \int_b^\infty y^{a-1}\ \ln(y)\ e^{-y} dy.$$

Proof. a) We have

$$H(X) = -\int_0^\infty f(x)\ \ln f(x) dx.$$

Now,

$$\begin{aligned}\int_0^\infty f(x)\ \ln f(x) dx =& \int_0^\infty f(x) \Bigg\{ \ln\left(\frac{\theta^2}{\eta^\delta + \theta k}\right) + \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j} \left(\frac{\eta^\delta \Gamma(\alpha)(\theta x)^{\beta-\alpha}}{k\theta\Gamma(\beta)}\right)^j \\ & + \ln\left(\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}\right) - \theta x \Bigg\} dx \\ =& A_1 + A_2 + A_3 - A_4, \end{aligned} \tag{12}$$

where

$$A_1 = \int_0^\infty \ln\left(\frac{\theta^2}{\eta^\delta + \theta k}\right) f(x) dx, \tag{13}$$

$$A_2 = \int_0^\infty f(x) \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j} \left(\frac{\eta^\delta \Gamma(\alpha)(\theta x)^{\beta-\alpha}}{k\theta\Gamma(\beta)}\right)^j dx, \tag{14}$$

$$A_3 = \int_0^\infty f(x) \ln\left(\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}\right) dx \tag{15}$$

and

$$A_4 = \int_0^\infty f(x) \theta x dx. \tag{16}$$

From (13)

$$\begin{aligned} A_1 &= \ln\left(\frac{\theta^2}{\eta^\delta + \theta k}\right) \int_0^\infty f(x) dx \\ &= \ln\left(\frac{\theta^2}{\eta^\delta + \theta k}\right). \end{aligned} \tag{17}$$

From (14)

$$\begin{aligned} A_2 =& \int_0^\infty \left(\frac{\theta^2}{\eta^\delta + \theta k}\right) \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j} \left(\frac{\eta^\delta \Gamma(\alpha)}{k\theta\Gamma(\beta)}\right)^j \\ & \times \left\{ \frac{k\Big(\theta x\Big)^{\alpha-1+j(\beta-\alpha)}}{\Gamma(\alpha)} + \frac{\eta^\delta \Big(\theta x\Big)^{\beta-1+j(\beta-\alpha)}}{\theta\Gamma(\beta)} \right\} e^{-\theta x} dx \\ =& \left(\frac{\theta^2}{\eta^\delta + \theta k}\right) \sum_{j=1}^\infty \frac{(-1)^{j+1}}{j} \left(\frac{\eta^\delta \Gamma(\alpha)}{k\theta\Gamma(\beta)}\right)^j \Bigg\{ \frac{k}{\Gamma(\alpha)} \int_0^\infty \Big(\theta x\Big)^{\alpha-1+j(\beta-\alpha)} e^{-\theta x} dx \\ & + \frac{\eta^\delta}{\theta\Gamma(\beta)} \int_0^\infty \Big(\theta x\Big)^{\beta-1+j(\beta-\alpha)} e^{-\theta x} dx \Bigg\} \end{aligned}$$

$$=\left(\frac{\theta^2}{\eta^\delta+\theta k}\right)\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}\left(\frac{\eta^\delta\Gamma(\alpha)}{k\theta\Gamma(\beta)}\right)^j\left\{\frac{k\Gamma\Big(\alpha+j(\beta-\alpha)\Big)}{\theta\Gamma(\alpha)}+\right.$$
$$\left.\frac{\eta^\delta\Gamma\Big(\beta+j(\beta-\alpha)\Big)}{\theta^2\Gamma(\beta)}\right\}. \tag{18}$$

From (15)

$$\begin{aligned}A_3=&\ln\left(\frac{k}{\Gamma(\alpha)}\right)\int_0^\infty f(x)dx+\frac{\theta^2}{(\eta^\delta+\theta k)}\left\{\frac{k(\alpha-1)}{\Gamma(\alpha)}\int_0^\infty\Big(\theta x\Big)^{\alpha-1}\ln\Big(\theta x\Big)e^{-\theta x}dx+\right.\\&\left.\frac{\eta^\delta(\alpha-1)}{\theta\Gamma(\beta)}\int_0^\infty\Big(\theta x\Big)^{\beta-1}\ln\Big(\theta x\Big)e^{-\theta x}dx\right\}\\=&\ln\left(\frac{k}{\Gamma(\alpha)}\right)+\frac{\theta^2}{(\eta^\delta+\theta k)}\left\{\frac{k(\alpha-1)}{\theta\Gamma(\alpha)}\Gamma'(\alpha)+\frac{\eta^\delta(\alpha-1)}{\theta^2\Gamma(\beta)}\Gamma'(\beta)\right\}\\=&\ln\left(\frac{k}{\Gamma(\alpha)}\right)+\frac{\theta^2}{(\eta^\delta+\theta k)}\left\{\frac{k(\alpha-1)}{\theta}\psi(\alpha)+\frac{\eta^\delta(\alpha-1)}{\theta^2}\psi(\beta)\right\}.\end{aligned}\tag{19}$$

From (16)

$$\begin{aligned}A_4&=\frac{\theta^2}{(\eta^\delta+\theta k)}\int_0^\infty\left\{\frac{k(\theta x)^\alpha}{\Gamma(\alpha)}+\frac{\eta^\delta(\theta x)^\beta}{\theta\Gamma(\beta)}\right\}e^{-\theta x}dx\\&=\frac{\theta^2}{(\eta^\delta+\theta k)}\left\{\frac{k\Gamma(\alpha+1)}{\theta\Gamma(\alpha)}+\frac{\eta^\delta\Gamma(\beta+1)}{\theta^2\Gamma(\beta)}\right\}\\&=\frac{\theta^2}{(\eta^\delta+\theta k)}\left\{\frac{k\alpha}{\theta}+\frac{\eta^\delta\beta}{\theta^2}\right\}.\end{aligned}\tag{20}$$

Substituting (17), (18), (19) and (20) in (12), we obtain (11).
b)

$$H^\nu(X)=\frac{1}{(1-\nu)}\ln\int_0^\infty f^\nu(x)dx;\ \text{for } \nu>0,\nu\neq1.$$

Now,

$$\begin{aligned}\int_0^\infty f^\nu(x)dx=&\left(\frac{\theta^2}{\eta^\delta+\theta k}\right)^\nu\int_0^\infty\sum_{j=0}^{\nu}\binom{\nu}{j}\left(\frac{\eta^\delta(\theta x)^{\beta-1}\Gamma(\alpha)}{\theta\Gamma(\beta)k(\theta x)^{\alpha-1}}\right)^j\\&\times\left(\frac{k(\theta x)^{\alpha-1}}{\Gamma(\alpha)}\right)^\nu e^{-\theta\nu x}dx\\=&\left(\frac{\theta^2}{\eta^\delta+\theta k}\right)^\nu\left(\frac{k}{\Gamma(\alpha)}\right)^\nu\sum_{j=0}^{\nu}\binom{\nu}{j}\left(\frac{\eta^\delta\Gamma(\alpha)}{\theta k\Gamma(\beta)}\right)^j\\&\times\int_0^\infty\Big(\theta x\Big)^{\nu(\alpha-1)+j(\beta-\alpha)}e^{-\theta\nu x}dx.\end{aligned}\tag{21}$$

Therefore,

$$H^{\nu}(X)=\frac{1}{1-\nu}\left\{\ln\left(\frac{k\theta^{\frac{2\nu-1}{\nu}}}{\nu^{\frac{(\alpha-1)\nu+1}{\nu}}(\eta^{\delta}+\theta k)\Gamma(\alpha)}\right)^{\nu}+\ln\left(\sum_{j=0}^{\nu}\binom{\nu}{j}\left(\frac{\eta^{\delta}\Gamma(\alpha)}{\theta k\Gamma(\beta)}\right)^{j}\frac{\Gamma\Big(\nu(\alpha-1)+j(\beta-\alpha)+1\Big)}{\nu^{j(\beta-\alpha)}}\right)\right\}.$$

c) We have

$$H^{\rho}(X)=\frac{1}{(\rho-1)}\left(1-\int_0^{\infty}f^{\rho}(x)dx\right);\ \text{for } \rho>0,\rho\neq 1.$$

From (21)

$$\int_0^{\infty}f^{\rho}(x)dx=\left(\frac{\theta^2}{\eta^{\delta}+\theta k}\right)^{\rho}\left(\frac{k}{\Gamma(\alpha)}\right)^{\rho}\sum_{j=0}^{\rho}\binom{\rho}{j}\left(\frac{\eta^{\delta}\Gamma(\alpha)}{\theta k\Gamma(\beta)}\right)^{j}\times\frac{\Gamma\Big(\rho(\alpha-1)+j(\beta-\alpha)+1\Big)}{\theta\rho^{(\alpha-1)\rho+j(\beta-\alpha)+1}}.$$

Therefore,

$$H^{\rho}(X)=\frac{1}{\rho-1}\left\{1-\left(\frac{k\theta^{\frac{2\rho-1}{\rho}}}{\rho^{\frac{(\alpha-1)\rho+1}{\rho}}(\eta^{\delta}+\theta k)\Gamma(\alpha)}\right)^{\rho}\times\sum_{j=0}^{\rho}\binom{\rho}{j}\left(\frac{\eta^{\delta}\Gamma(\alpha)}{\theta k\Gamma(\beta)}\right)^{j}\frac{\Gamma\Big(\rho(\alpha-1)+j(\beta-\alpha)+1\Big)}{\rho^{j(\beta-\alpha)}}\right\}.$$

d) The proof is same as that of (a) and hence omitted. ■

6. Estimation of Parameters

For estimating the parameters of *EVGLD*, we discuss two methods, namely the method of moments and the method of maximum likelihood estimation.

6.1. Method of Moment Estimation

From (6), the r^{th} raw moment about origin for the *EVGL* random variable is given by

$$\mu_r^{'}=\frac{1}{\eta^{\delta}+\theta k}\left\{\frac{k\Gamma(\alpha+r)}{\theta^{r-1}\Gamma(\alpha)}+\frac{\eta^{\delta}\Gamma(\beta+r)}{\theta^{r}\Gamma(\beta)}\right\}.$$

Setting $r=1,2,3,4,5$, and 6, the first six raw moments are obtained. Equating these raw moments to the corresponding sample moments; say $m_1^{'}$, $m_2^{'}$, $m_3^{'}$, $m_4^{'}$, $m_5^{'}$ and $m_6^{'}$, where $m_r^{'}=\frac{1}{n}\sum_{i=1}^{n}X_i^r$, we get

$$\frac{1}{\theta(\eta^{\delta}+\theta k)}\left\{\alpha\theta k+\beta\eta^{\delta}\right\}=m_1^{'},$$

$$\frac{1}{\theta^2(\eta^\delta+\theta k)}\left\{\alpha(\alpha+1)\theta k+\beta(\beta+1)\eta^\delta\right\}=m_2^{'},$$

$$\frac{1}{\theta^3(\eta^\delta+\theta k)}\left\{\alpha(\alpha+1)(\alpha+2)\theta k+\beta(\beta+1)(\beta+2)\eta^\delta\right\}=m_3^{'},$$

$$\frac{1}{\theta^4(\eta^\delta+\theta k)}\Big\{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\theta k+$$
$$\beta(\beta+1)(\beta+2)(\beta+3)\eta^\delta\Big\}=m_4^{'},$$

$$\frac{1}{\theta^5(\eta^\delta+\theta k)}\Big\{\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)\theta k+$$
$$\beta(\beta+1)(\beta+2)(\beta+3)(\beta+4)\eta^\delta\Big\}=m_5^{'}$$

and

$$\frac{1}{\theta^6(\eta^\delta+\theta k)}\Big\{\alpha(\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)\theta k+$$
$$\beta(\beta+1)(\beta+2)(\beta+3)(\beta+4)(\beta+5)\eta^\delta\Big\}=m_6^{'}.$$

Since the above system of equations is non-linear, the numerical solutions are obtained using iterative procedures. From a practical point of view, we consider two real life data sets which are given in Section 9 and find the first six sample moments. These moments are equated to the corresponding moments of the population and, solving this system of equations iteratively using statistical software like MATHCAD, MATHEMATICA and R, we get the moment estimators of the parameters of EVGLD which are given in Table 1 and Table 2.

6.2. Method of Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $EVGLD$ with unknown parameter vector $\Theta = \left(\theta, \alpha, \beta, k, \eta, \delta\right)$. The likelihood function for Θ is

$$\begin{aligned} l(\Theta) &= \prod_{i=1}^{n} f_i\left(x; \theta, \alpha, \beta, k, \eta, \delta\right) \\ &= \left(\frac{\theta^2}{\eta^\delta+\theta k}\right)^n e^{-\theta\sum_{i=1}^{n} x_i}\Big(\Gamma(\alpha)\Gamma(\beta)\Big)^{-n}\prod_{i=1}^{n}\Big(\Gamma(\beta)k\theta^{\alpha-1}x_i^{\alpha-1}+\Gamma(\alpha)\eta^\delta\theta^{\beta-2}x_i^{\beta-1}\Big). \end{aligned}$$

The partial derivatives of $\ln l(\Theta)$ with respect to the parameters are

$$\frac{\partial \ln l}{\partial \theta} = \frac{2n}{\theta} - \frac{nk}{(\eta^\delta+\theta k)} - \sum_{i=1}^{n} x_i +$$
$$\sum_{i=1}^{n}\left(\frac{k\Gamma(\beta)(\alpha-1)\theta^{\alpha-2}x_i^{\alpha-1}+\eta^\delta\Gamma(\alpha)(\beta-2)\theta^{\beta-3}x_i^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1}+\eta^\delta\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}}\right),$$

$$\frac{\partial \ln l}{\partial \alpha} = -n\psi(\alpha) + \sum_{i=1}^{n}\left(\frac{k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1}\ln(\theta x_i)+\eta^\delta\Gamma^{'}(\alpha)\theta^{\beta-2}x_i^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1}+\eta^\delta\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}}\right),$$

$$\frac{\partial \ln l}{\partial \beta} = -n\psi(\beta) + \sum_{i=1}^{n}\left(\frac{k\Gamma'(\beta)\theta^{\alpha-1}x_i^{\alpha-1} + \eta^{\delta}\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}\ln(\theta x_i)}{k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1} + \eta^{\delta}\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}}\right),$$

$$\frac{\partial \ln l}{\partial k} = \frac{-n\theta}{(\eta^{\delta}+\theta k)} + \sum_{i=1}^{n}\left(\frac{\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1}}{k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1} + \eta^{\delta}\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}}\right),$$

$$\frac{\partial \ln l}{\partial \eta} = \frac{-n\delta\eta^{\delta-1}}{(\eta^{\delta}+\theta k)} + \sum_{i=1}^{n}\left(\frac{\delta\eta^{\delta-1}\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1} + \eta^{\delta}\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}}\right)$$

and

$$\frac{\partial \ln l}{\partial \delta} = \frac{-n\eta^{\delta}\ \ln(\eta)}{(\eta^{\delta}+\theta k)} + \sum_{i=1}^{n}\left(\frac{\eta^{\delta}\ \ln(\eta)\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}}{k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1} + \eta^{\delta}\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1}}\right).$$

As in the case of solving moment equations, the above non-linear system of likelihood equations are solved iteratively. Here we first set moment estimates obtained from the real life data sets as the initial solution of the parameters and then solving the above non-linear system of likelihood equations (using statistical software like MATHCAD, MATHEMATICA and R) we get the maximum likelihood estimates of the parameters of EVGLD which are given in the tables in the last section. From the tables, it is observed that chi-square statistic for the EVGLD is lower than those of competing models showing that our model satisfactorily fits better for these data sets.

7. Fisher Information Matrix

The second and cross partial derivatives with respect to the parameters are

$$\frac{\partial^2 \ln l}{\partial \theta^2} = \frac{-2n}{\theta^2} + \frac{nk^2}{(\eta^{\delta}+\theta k)^2} + \frac{1}{\theta^2}\sum_{i=1}^{n}\frac{1}{(A_i+B_i)^2}\Big\{\Big(A_i+B_i\Big)\Big((\alpha-1)(\alpha-2)A_i + (\beta-2)(\beta-3)B_i\Big) - \Big((\alpha-1)A_i + (\beta-2)B_i\Big)^2\Big\},$$

$$\frac{\partial^2 \ln l}{\partial \theta \partial \alpha} = \frac{1}{\theta}\sum_{i=1}^{n}\frac{1}{(A_i+B_i)^2}\Big\{\Big(A_i+B_i\Big)\Big(\Big\{(\alpha-1)\ \ln(\theta x_i)+1\Big\}A_i + (\beta-2)\psi(\alpha)B_i\Big) - \Big((\alpha-1)A_i+(\beta-2)B_i\Big)\Big(\ln(\theta x_i)A_i + \psi(\alpha)B_i\Big)\Big\},$$

$$\frac{\partial^2 \ln l}{\partial \theta \partial \beta} = \frac{1}{\theta}\sum_{i=1}^{n}\frac{1}{(A_i+B_i)^2}\Big\{\Big(A_i+B_i\Big)\Big((\alpha-1)\psi(\beta)A_i + \Big\{(\beta-2)\ \ln(\theta x_i)+1\Big\}B_i\Big) - \Big((\alpha-1)A_i+(\beta-2)B_i\Big)\Big(\psi(\beta)A_i + \ln(\theta x_i)B_i\Big)\Big\},$$

$$\frac{\partial^2 \ln l}{\partial \theta \partial k} = \frac{-n\eta^{\delta}}{(\eta^{\delta}+\theta k)^2} + \frac{\alpha-\beta+1}{k\theta}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2},$$

$$\frac{\partial^2 \ln l}{\partial \theta \partial \eta} = \frac{nk\delta\eta^{\delta-1}}{(\eta^{\delta}+\theta k)^2} + \frac{\delta\Big(\beta-\alpha-1\Big)}{\eta\theta}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2},$$

$$\frac{\partial^2 \ln l}{\partial\theta\partial\delta} = \frac{nk\eta^{\delta}\ \ln(\eta)}{(\eta^{\delta}+\theta k)^2} + \frac{\ln(\eta)\Big(\beta-\alpha-1\Big)}{\theta}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2},$$

$$\frac{\partial^2 \ln l}{\partial\alpha^2} = -n\psi^{'}(\alpha)+\sum_{i=1}^{n}\frac{1}{(A_i+B_i)^2}\left\{\Big(A_i+B_i\Big)\left(A_i\Big(\ln(\theta x_i)\Big)^2+\frac{\Gamma^{''}(\alpha)}{\Gamma(\alpha)}B_i\right)-\left(A_i\ \ln(\theta x_i)+\frac{\Gamma^{'}(\alpha)}{\Gamma(\alpha)}B_i\right)^2\right\},$$

$$\frac{\partial^2 \ln l}{\partial\alpha\partial\beta} = \sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2}\Big\{\ln(\theta x_i)\Big(\psi(\alpha)+\psi(\beta)-\ln(\theta x_i)\Big)-\psi(\alpha)\psi(\beta)\Big\},$$

$$\frac{\partial^2 \ln l}{\partial\alpha\partial k} = \frac{1}{k}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2}\Big\{\ln(\theta x_i)-\psi(\alpha)\Big\},$$

$$\frac{\partial^2 \ln l}{\partial\alpha\partial\eta} = \frac{\delta}{\eta}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2}\Big\{\psi(\alpha)-\ln(\theta x_i)\Big\},$$

$$\frac{\partial^2 \ln l}{\partial\alpha\partial\delta} = \ln(\eta)\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2}\Big\{\psi(\alpha)-\ln(\theta x_i)\Big\},$$

$$\frac{\partial^2 \ln l}{\partial\beta^2} = -n\psi^{'}(\beta)+\sum_{i=1}^{n}\frac{1}{(A_i+B_i)^2}\left\{\Big(A_i+B_i\Big)\left(\frac{\Gamma^{''}(\beta)}{\Gamma(\beta)}A_i+B_i\Big(\ln(\theta x_i)\Big)^2\right)-\left(\frac{\Gamma^{'}(\beta)}{\Gamma(\beta)}A_i+B_i\ln(\theta x_i)\right)^2\right\},$$

$$\frac{\partial^2 \ln l}{\partial\beta\partial k} = \frac{1}{k}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2}\Big\{\psi(\beta)-\ln(\theta x_i)\Big\},$$

$$\frac{\partial^2 \ln l}{\partial\beta\partial\eta} = \frac{\delta}{\eta}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2}\Big\{\ln(\theta x_i)-\psi(\beta)\Big\},$$

$$\frac{\partial^2 \ln l}{\partial\beta\partial\delta} = \ln(\eta)\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2}\Big\{\ln(\theta x_i)-\psi(\beta)\Big\},$$

$$\frac{\partial^2 \ln l}{\partial k^2} = \frac{n\theta^2}{(\eta^{\delta}+\theta k)^2}-\frac{1}{k^2}\sum_{i=1}^{n}\frac{A_i^2}{(A_i+B_i)^2},$$

$$\frac{\partial^2 \ln l}{\partial k\partial\eta} = \frac{n\theta\delta\eta^{\delta-1}}{(\eta^{\delta}+\theta k)^2}-\frac{\delta}{\eta k}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2},$$

$$\frac{\partial^2 \ln l}{\partial k\partial\delta} = \frac{n\theta\eta^{\delta}\ \ln(\eta)}{(\eta^{\delta}+\theta k)^2}-\frac{\ln(\eta)}{k}\sum_{i=1}^{n}\frac{A_iB_i}{(A_i+B_i)^2},$$

$$\frac{\partial^2 \ln l}{\partial\eta^2} = \frac{-n\delta\eta^{\delta-2}}{(\eta^{\delta}+\theta k)^2}\Big((\delta-1)\theta k-\eta^{\delta}\Big)+\frac{1}{\eta^2}\sum_{i=1}^{n}\frac{1}{(A_i+B_i)^2}\Big\{\Big(A_i+B_i\Big)B_i\delta(\delta-1)-(\delta B_i)^2\Big\},$$

$$\frac{\partial^2 \ln l}{\partial\eta\partial\delta} = \frac{-n}{(\eta^\delta+\theta k)^2}\Big\{\theta k\eta^{\delta-1}\Big(\delta\ \ln(\eta)+1\Big)+\eta^{2\delta-1}\Big\}+ \frac{1}{\eta}\sum_{i=1}^{n}\frac{A_iB_i\Big(1+\delta\ \ln(\eta)\Big)+B_i^2}{(A_i+B_i)^2}$$

and

$$\frac{\partial^2 \ln l}{\partial\delta^2} = \frac{-n\theta k\eta^\delta\Big(\ln(\eta)\Big)^2}{(\eta^\delta+\theta k)^2}+\sum_{i=1}^{n}\frac{A_iB_i\Big(\ln(\eta)\Big)^2}{(A_i+B_i)^2},$$

where

$$A_i = k\Gamma(\beta)\theta^{\alpha-1}x_i^{\alpha-1},$$

$$B_i = \eta^\delta\Gamma(\alpha)\theta^{\beta-2}x_i^{\beta-1},$$

$$\psi'(a) = \frac{\Gamma(a)\Gamma''(a)-\Big(\Gamma'(a)\Big)^2}{\Gamma^2(a)}$$

is the polygamma function and

$$\Gamma^{(n)}(a) = \int_0^\infty t^{a-1}\Big(\ln(t)\Big)^n e^{-t}dt$$

is the n^{th} order derivative of gamma function.

The expected Fisher information matrix for the $EVGLD$ is

$$I(\Theta) = \begin{pmatrix} -E\left[\frac{\partial^2\ln l}{\partial\theta^2}\right] & -E\left[\frac{\partial^2\ln l}{\partial\theta\partial\alpha}\right] & -E\left[\frac{\partial^2\ln l}{\partial\theta\partial\beta}\right] & -E\left[\frac{\partial^2\ln l}{\partial\theta\partial k}\right] & -E\left[\frac{\partial^2\ln l}{\partial\theta\partial\eta}\right] & -E\left[\frac{\partial^2\ln l}{\partial\theta\partial\delta}\right] \\ -E\left[\frac{\partial^2\ln l}{\partial\alpha\partial\theta}\right] & -E\left[\frac{\partial^2\ln l}{\partial\alpha^2}\right] & -E\left[\frac{\partial^2\ln l}{\partial\alpha\partial\beta}\right] & -E\left[\frac{\partial^2\ln l}{\partial\alpha\partial k}\right] & -E\left[\frac{\partial^2\ln l}{\partial\alpha\partial\eta}\right] & -E\left[\frac{\partial^2\ln l}{\partial\alpha\partial\delta}\right] \\ -E\left[\frac{\partial^2\ln l}{\partial\beta\partial\theta}\right] & -E\left[\frac{\partial^2\ln l}{\partial\beta\partial\alpha}\right] & -E\left[\frac{\partial^2\ln l}{\partial\beta^2}\right] & -E\left[\frac{\partial^2\ln l}{\partial\beta\partial k}\right] & -E\left[\frac{\partial^2\ln l}{\partial\beta\partial\eta}\right] & -E\left[\frac{\partial^2\ln l}{\partial\beta\partial\delta}\right] \\ -E\left[\frac{\partial^2\ln l}{\partial k\partial\theta}\right] & -E\left[\frac{\partial^2\ln l}{\partial k\partial\alpha}\right] & -E\left[\frac{\partial^2\ln l}{\partial k\partial\beta}\right] & -E\left[\frac{\partial^2\ln l}{\partial k^2}\right] & -E\left[\frac{\partial^2\ln l}{\partial k\partial\eta}\right] & -E\left[\frac{\partial^2\ln l}{\partial k\partial\delta}\right] \\ -E\left[\frac{\partial^2\ln l}{\partial\eta\partial\theta}\right] & -E\left[\frac{\partial^2\ln l}{\partial\eta\partial\alpha}\right] & -E\left[\frac{\partial^2\ln l}{\partial\eta\partial\beta}\right] & -E\left[\frac{\partial^2\ln l}{\partial\eta\partial k}\right] & -E\left[\frac{\partial^2\ln l}{\partial\eta^2}\right] & -E\left[\frac{\partial^2\ln l}{\partial\eta\partial\delta}\right] \\ -E\left[\frac{\partial^2\ln l}{\partial\delta\partial\theta}\right] & -E\left[\frac{\partial^2\ln l}{\partial\delta\partial\alpha}\right] & -E\left[\frac{\partial^2\ln l}{\partial\delta\partial\beta}\right] & -E\left[\frac{\partial^2\ln l}{\partial\delta\partial k}\right] & -E\left[\frac{\partial^2\ln l}{\partial\delta\partial\eta}\right] & -E\left[\frac{\partial^2\ln l}{\partial\delta^2}\right] \end{pmatrix}.$$

The expected Fisher information matrix can be approximated by the observed Fisher information matrix $J(\widehat{\Theta})$ given by

$$J(\widehat{\Theta}) = \begin{pmatrix} -\frac{\partial^2\ln l}{\partial\theta^2} & -\frac{\partial^2\ln l}{\partial\theta\partial\alpha} & -\frac{\partial^2\ln l}{\partial\theta\partial\beta} & -\frac{\partial^2\ln l}{\partial\theta\partial k} & -\frac{\partial^2\ln l}{\partial\theta\partial\eta} & -\frac{\partial^2\ln l}{\partial\theta\partial\delta} \\ -\frac{\partial^2\ln l}{\partial\alpha\partial\theta} & -\frac{\partial^2\ln l}{\partial\alpha^2} & -\frac{\partial^2\ln l}{\partial\alpha\partial\beta} & -\frac{\partial^2\ln l}{\partial\alpha\partial k} & -\frac{\partial^2\ln l}{\partial\alpha\partial\eta} & -\frac{\partial^2\ln l}{\partial\alpha\partial\delta} \\ -\frac{\partial^2\ln l}{\partial\beta\partial\theta} & -\frac{\partial^2\ln l}{\partial\beta\partial\alpha} & -\frac{\partial^2\ln l}{\partial\beta^2} & -\frac{\partial^2\ln l}{\partial\beta\partial k} & -\frac{\partial^2\ln l}{\partial\beta\partial\eta} & -\frac{\partial^2\ln l}{\partial\beta\partial\delta} \\ -\frac{\partial^2\ln l}{\partial k\partial\theta} & -\frac{\partial^2\ln l}{\partial k\partial\alpha} & -\frac{\partial^2\ln l}{\partial k\partial\beta} & -\frac{\partial^2\ln l}{\partial k^2} & -\frac{\partial^2\ln l}{\partial k\partial\eta} & -\frac{\partial^2\ln l}{\partial k\partial\delta} \\ -\frac{\partial^2\ln l}{\partial\eta\partial\theta} & -\frac{\partial^2\ln l}{\partial\eta\partial\alpha} & -\frac{\partial^2\ln l}{\partial\eta\partial\beta} & -\frac{\partial^2\ln l}{\partial\eta\partial k} & -\frac{\partial^2\ln l}{\partial\eta^2} & -\frac{\partial^2\ln l}{\partial\eta\partial\delta} \\ -\frac{\partial^2\ln l}{\partial\delta\partial\theta} & -\frac{\partial^2\ln l}{\partial\delta\partial\alpha} & -\frac{\partial^2\ln l}{\partial\delta\partial\beta} & -\frac{\partial^2\ln l}{\partial\delta\partial k} & -\frac{\partial^2\ln l}{\partial\delta\partial\eta} & -\frac{\partial^2\ln l}{\partial\delta^2} \end{pmatrix}.$$

That is,

$$\lim_{n\to\infty} \frac{1}{n} J(\widehat{\Theta}) = I(\Theta).$$

For large n, the following approximation can be used.

$$J(\widehat{\Theta}) = nI(\Theta).$$

8. Asymptotic Confidence Interval

In this section, we present the asymptotic confidence intervals for the parameters of the $EVGLD$. Let $\widehat{\Theta} = \left(\widehat{\theta}, \widehat{\alpha}, \widehat{\beta}, \widehat{k}, \widehat{\eta}, \widehat{\delta}\right)$ be the maximum likelihood estimator of $\Theta = \left(\theta, \alpha, \beta, k, \eta, \delta\right)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have $\sqrt{n}\left(\Theta - \widehat{\Theta}\right) \xrightarrow{d} N_6\left(\underline{0}, I^{-1}(\Theta)\right)$, where $I(\Theta)$ is the expected Fisher information matrix. The asymptotic behaviour is still valid if $I(\Theta)$ is replaced by the observed Fisher information matrix, $J(\widehat{\Theta})$. The multivariate normal distribution, $N_6\left(\underline{0}, I^{-1}(\Theta)\right)$ with mean vector $\underline{0} = \left(0,0,0,0,0,0\right)^{\tau}$ can be used to construct confidence intervals for the model parameters. The approximate $100(1-\phi)\%$ two-sided confidence intervals for $\theta, \alpha, \beta, k, \eta, \delta$ are $\widehat{\theta} \pm Z_{\frac{\phi}{2}} \sqrt{I^{-1}_{\theta\theta}(\widehat{\Theta})}$, $\widehat{\alpha} \pm Z_{\frac{\phi}{2}} \sqrt{I^{-1}_{\alpha\alpha}(\widehat{\Theta})}$, $\widehat{\beta} \pm Z_{\frac{\phi}{2}} \sqrt{I^{-1}_{\beta\beta}(\widehat{\Theta})}$, $\widehat{k} \pm Z_{\frac{\phi}{2}} \sqrt{I^{-1}_{kk}(\widehat{\Theta})}$, $\widehat{\eta} \pm Z_{\frac{\phi}{2}} \sqrt{I^{-1}_{\eta\eta}(\widehat{\Theta})}$ and $\widehat{\delta} \pm Z_{\frac{\phi}{2}} \sqrt{I^{-1}_{\delta\delta}(\widehat{\Theta})}$ respectively, where $I^{-1}_{\theta\theta}(\widehat{\Theta})$, $I^{-1}_{\alpha\alpha}(\widehat{\Theta})$, $I^{-1}_{\beta\beta}(\widehat{\Theta})$, $I^{-1}_{kk}(\widehat{\Theta})$, $I^{-1}_{\eta\eta}(\widehat{\Theta})$ and $I^{-1}_{\delta\delta}(\widehat{\Theta})$ are diagonal elements of $J^{-1}(\widehat{\Theta})$ and $Z_{\frac{\phi}{2}}$ is the upper $\frac{\phi}{2}^{\text{th}}$ percentile of a standard normal distribution.

9. Simulation

In this section, we demonstrate the applicability of the $EVGL$ model for two real data sets. The data represents the survival times of guinea pigs injected with different doses of tubercle bacilli (see, Bjerkedal, 1960). Guinea pigs are known to have high susceptibility to human tuberculosis. Even an infection initiated with a few virulent tubercle bacilli will lead to progressive disease and death. We used the data sets obtained under the regimen 4.3 and regimen 6.6. There were 72 observations corresponding to the regimen 4.3 and regimen 6.6 and are respectively given by

Survival times of 72 guinea pigs under regimen 4.3
0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Survival times of 72 guinea pigs under regimen 6.6

0.12, 0.15, 0.22, 0.24, 0.24, 0.32, 0.32, 0.33, 0.34, 0.38, 0.38,0.43, 0.44, 0.48, 0.52, 0.53, 0.54, 0.54, 0.55, 0.56, 0.57, 0.58, 0.58, 0.59, 0.60, 0.60, 0.60, 0.60, 0.61, 0.62, 0.63, 0.65, 0.65, 0.67, 0.68, 0.70, 0.70, 0.72, 0.73, 0.75, 76, 0.76, 0.81, 0.83, 0.84, 0.85, 0.87, 0.91, 0.95, 0.96, 0.98, 0.99, 1.09, 1.10, 1.21, 1.27, 1.29, 1.31, 1.43, 1.46, 1.46, 1.75, 1.75, 2.11, 2.33, 2.58, 2.58, 2.63, 2.97, 3.41, 3.41, 3.76.

For these data sets we fit the proposed $EVGLD$ and its sub models LD_1, LD_2, GLD_1 and GLD_2. Moment estimators and maximum likelihood estimators of the parameters of the models and their χ^2 values are calculated and given in Tables 1 and 2. They indicate that $EVGLD$ fits the two data sets better than the other distributions.

Figure 7 shows the survival function of the guinea pigs under regimen 4.3 and 6.6. From the figure, we can say that the survival function decreases with increase in the numbers of bacilli in the challenge dose. The vitality function of the guinea pigs under regimen 4.3 and 6.6 are given in Figure 8. It is seen that the vitality function of guinea pigs under regimen 4.3 is greater compared with regimen 6.6. From this we can say that the average life span of the guinea pigs under regimen 4.3 whose age exceeds x is greater compared with those under regimen 6.6.

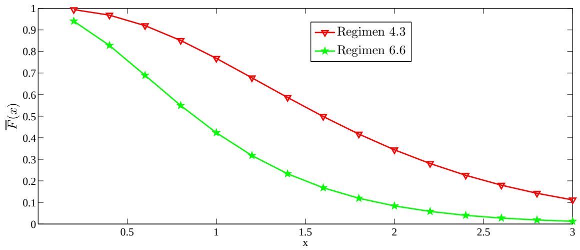

Figure 7: Plots of survival function of guinea pigs under regimen 4.3 and 6.6.

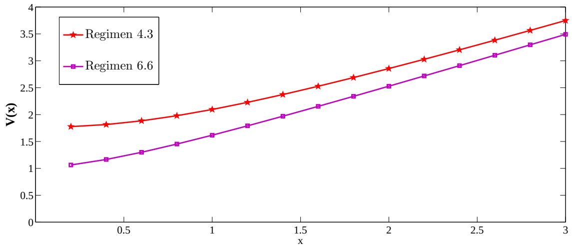

Figure 8: Plots of vitality function of guinea pigs under regimen 4.3 and 6.6.

Table 1: Comparison of fit of *EVGLD* using different methods of estimation of 72 guinea pigs under regimen 4.3.

Count	*Observed*	Expected frequency by method of moments					Expected frequency by MLE				
		LD_1	LD_2	GLD_1	GLD_2	$EVGLD$	LD_1	LD_2	GLD_1	GLD_2	$EVGLD$
0-0.5	3	15	16	20	4	3	10	9	11	7	4
0.5-1.0	10	13	14	15	14	12	12	13	15	11	11
1.0-1.25	17	6	6	6	8	8	8	9	8	13	8
1.25-1.5	6	5	6	5	8	8	5	4	7	6	9
1.5-1.75	8	5	5	4	7	7	5	6	4	5	8
1.75-2.0	5	4	4	4	6	7	4	4	5	2	7
2.0-2.25	5	4	4	3	5	6	4	3	2	4	5
2.25-2.5	4	3	3	3	5	5	4	2	2	7	5
2.5-3	6	5	5	4	6	7	9	9	9	4	6
3-4	4	6	5	5	6	6	5	6	4	8	6
4-5	3	3	2	2	2	2	4	5	3	3	2
5-∞	1	3	2	1	1	1	2	2	2	2	1
Total	72	72	72	72	72	72	72	72	72	72	72
df		7	5	3	6	2	7	5	3	6	2
Estimated values of parameters		$\widehat{\theta}$=0.868	$\widehat{\alpha}$= 1.3 $\widehat{\theta}$=1	$\widehat{\alpha}$=0.9 $\widehat{\gamma}$= 0.9 $\widehat{\theta}$=1	$\widehat{\alpha}$=3.85 $\widehat{\theta}$=1.81	$\widehat{\alpha}$= 3.8 $\widehat{\theta}$=1.9 $\widehat{\beta}$=2.59 $\widehat{k}$=2.1 $\widehat{\eta}$=1.85 $\widehat{\delta}$=0.65	$\widehat{\theta}$=0.9	$\widehat{\alpha}$=1.5 $\widehat{\theta}$=0.95	$\widehat{\alpha}$=0.86 $\widehat{\gamma}$=0.91 $\widehat{\theta}$=0.90	$\widehat{\alpha}$= 3.9 $\widehat{\theta}$=1.8	$\widehat{\alpha}$=3.8 $\widehat{\theta}$=1.97 $\widehat{\beta}$=2.59 $\widehat{k}$=2.09 $\widehat{\eta}$=1.828 $\widehat{\delta}$=0.655
χ^2		34.81	34.05	41.44	12.7	12.2	18.63	17.67	22.75	10.98	9.22

Table 2: Comparison of fit of *EVGLD* using different methods of estimation of 72 guinea pigs under regimen 6.6.

Count	*Observed*	Expected frequency by method of moments					Expected frequency by MLE				
		LD_1	LD_2	GLD_1	GLD_2	$EVGLD$	LD_1	LD_2	GLD_1	GLD_2	$EVGLD$
0-0.25	5	14	14	28	13	8	8	13	25	10	6
0.25-0.5	9	12	12	11	12	11	9	10	10	13	8
0.5-0.65	19	6	7	5	7	7	9	14	9	12	15
0.65-0.75	7	4	4	3	4	5	7	5	5	6	7
0.75-0.85	6	4	3	3	4	4	8	5	4	3	5
0.85-1.0	6	5	5	3	5	6	7	6	2	4	7
1.0-1.25	3	7	7	4	7	8	11	4	3	6	4
1.25-1.5	6	5	5	3	5	6	3	5	4	5	8
1.5-1.75	2	4	4	3	4	5	1	2	2	3	3
1.75-2.5	2	7	7	5	7	8	5	4	4	6	4
2.5-3	4	2	2	2	2	2	3	3	2	2	3
3-4	3	2	2	2	2	2	1	1	2	2	2
Total	72	72	72	72	72	72	72	72	72	72	72
df		6	5	3	5	2	6	5	3	5	2
Estimated values of parameters		$\widehat{\theta}$=1.42	$\widehat{\alpha}$= 1.21 $\widehat{\theta}$=1.51	$\widehat{\alpha}$=0.5 $\widehat{\gamma}$= 0.5 $\widehat{\theta}$=1.1	$\widehat{\alpha}$=2.1 $\widehat{\theta}$=1.51	$\widehat{\alpha}$= 1.2 $\widehat{\theta}$=2 $\widehat{\beta}$=2.4 $\widehat{k}$=2.35 $\widehat{\eta}$=2.1 $\widehat{\delta}$=3.5	$\widehat{\theta}$=1.512	$\widehat{\alpha}$=1.41 $\widehat{\theta}$=1.42	$\widehat{\alpha}$=0.54 $\widehat{\gamma}$=0.6 $\widehat{\theta}$=0.9	$\widehat{\alpha}$= 1.98 $\widehat{\theta}$=1.62	$\widehat{\alpha}$=1.22 $\widehat{\theta}$=2.46 $\widehat{\beta}$=2.65 $\widehat{k}$=2.87 $\widehat{\eta}$=2.81 $\widehat{\delta}$=3.8
χ^2		41.58	36.00	67.86	33.12	28.94	36.09	28.31	54.32	24.95	22.41

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