

# BETA SLASHED GENERALISED HALF-NORMAL DISTRIBUTION

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**Key words:** Beta distribution, Characteristic function, Half-normal distribution, Skewness, Stochastic representation.

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**Abstract:** In this paper we propose the beta slashed generalised half-normal distribution, which includes some important distributions such as the half-normal, slashed half-normal and generalised slashed half-normal distributions. Explicit expressions for the cumulative distribution and characteristic functions are derived. The maximum likelihood estimates of the parameters are obtained via the EM algorithm and the value of the proposed model is illustrated with an application on fatigue data.

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## 1. Introduction

The folded normal distribution and its special case, the half normal (HN) distribution, arises in situations where one is interested in the magnitude of a normally distributed random variable (r.v.). Cooray and Ananda (2008) defined the generalised half-normal (GHN) distribution by extending the well-known half-normal distribution to model static fatigue data. They derived the model from the relationship between *static fatigue* and the *failure time* of a certain component. They named the distribution the GHN, due to the resemblance to the cumulative distribution function (cdf) of the HN distribution. Their definition of the GHN is given below.

**Definition 1** (Cooray and Ananda, 2008). The r.v.  $X$  is said to have a GHN distribution with scale parameter  $\sigma \in \mathbb{R}^+$  and shape parameter  $\alpha \in \mathbb{R}^+$ , written  $X \sim GHN(\sigma, \alpha)$ , if the density function is given by

$$f_X(x) = \frac{2\alpha}{\sigma^\alpha} x^{\alpha-1} \phi\left(\left(\frac{x}{\sigma}\right)^\alpha\right), \quad x > 0$$

with  $\phi$  the standard normal density function.

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Setting  $\alpha = 1$  in Definition 1, we obtain the density function of the HN distribution.

Olmos, Varela, Gómez and Bolfarine (2012) extended the half-normal distribution by compounding it with a uniform distribution, calling the resulting distribution the slashed half-normal (SHN) distribution. They also applied this approach to the GHN distribution, resulting in the slashed generalised half normal (SGHN) distribution as defined below.

**Definition 2** (Olmos, Varela, Bolfarine and Gómez, 2014) The r.v.  $Y$  is said to have a SGHN distribution with scale parameter  $\sigma \in \mathbb{R}^+$  and shape parameters  $\alpha, q \in \mathbb{R}^+$ , written  $Y \sim SGHN(\sigma, \alpha, q)$ , if the density function is given by

$$f_Y(y) = cy^{-(q+1)}G\left(y^{2\alpha}, \frac{q+\alpha}{2\alpha}, \frac{1}{2\alpha^{2\alpha}}\right), \quad y > 0$$

with

$$G(y; a, b) = \frac{b^a}{\Gamma(a)} \int_0^y u^{a-1} e^{-bu} du$$

the cdf of a gamma distribution and

$$c = q \sqrt{\frac{2^{\frac{q}{\alpha}}}{\pi}} \sigma^q \Gamma\left(\frac{q+\alpha}{2\alpha}\right)$$

the normalizing constant.

Taking  $\alpha = 1$  in Definition 2 with  $q$  approaching infinity yields the half-normal distribution, however as  $q$  goes to zero the result of Olmos et al. (2012) is obtained. In what follows we extend the result of the latter reference to a more general case by compounding the GHN distribution with a beta distribution. It will be shown that the resulting distribution covers all previously proposed distributions.

This paper is organized as follows: In Section 2, we define the beta slashed generalised half-normal (BSGHN) distribution and derive the hypergeometric and stochastic representations of the distribution. Section 3 presents some properties of the newly defined distribution, while the estimation of the parameters of the distribution is considered in Section 4. A simulation study is considered in Section 5 followed by a real data application in Section 6.

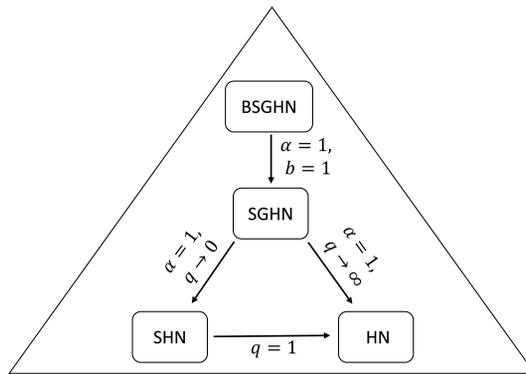
## 2. Beta Slashed Generalised Half-normal Distribution

In this section we define the beta slashed generalised half normal (BSGHN) distribution in series form. A stochastic representation is also derived followed by the derivation of the hypergeometric representation.

**Definition 3** The r.v.  $Z$  is said to have a BSGHN distribution with scale parameter  $\sigma \in \mathbb{R}^+$  and shape parameters  $\alpha, q, a, b \in \mathbb{R}^+$ , written  $Z \sim BSGHN(\sigma, \alpha, q, a, b)$ , if the density function is given by

$$f_Z(z) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\sigma^\alpha B(a, b)} \sum_{i=0}^{\infty} \left(\frac{-1}{2\sigma^{2\alpha}}\right)^i \frac{B(c_i, b)}{\Gamma(i+1)} z^{\alpha(2i+1)-1}, \quad z > 0,$$

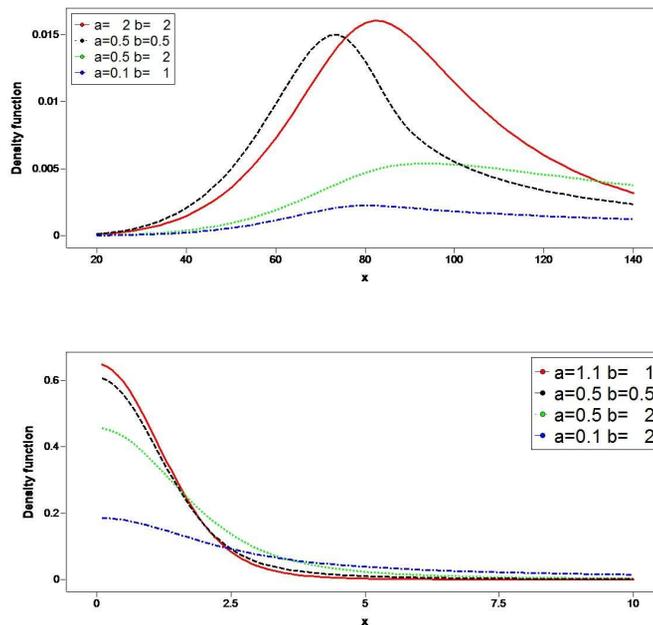
and  $c_i = \alpha(2i+1) + q(a-1) + 1$ .



**Figure 1:** Branching family.

The BSGHN distribution is a branching family as depicted in Figure 1.

Figure 2 present graphs of the density function of the BSGHN distribution for selected parameter values, demonstrating the effect of the parameters moving from a truncated shape to a unimodal shape.



**Figure 2:** Density function of the *BSGHN* distribution for selected parameter values Unimodal case  $\sigma = 70, \alpha = 5, q = 2$ ; Truncated case  $\sigma = 1, \alpha = 1, q = 4$ .

A stochastic representation for the BSGHN is given by the following theorem.

**Theorem 1** Let  $Z$  be a positive r.v. and  $\sigma, \alpha, q, a, b \in \mathbb{R}^+$ .

Then  $Z \sim BSGHN(\sigma, \alpha, q, a, b)$  if and only if  $Z = RX$ , with  $X \sim GHN(\sigma, \alpha)$  and if the density function of  $R$  is given by

$$f_R(r) = \frac{1}{qB(a, b)} r^{\frac{a}{q}-1} \left(1 - r^{\frac{1}{q}}\right)^{b-1}, \quad 0 < r < 1.$$

**Proof.** Let  $Y$  be a beta distributed random variable with shape parameters  $a$  and  $b$ ,  $Y \sim B(a, b)$ .

Firstly we show that the r.v.  $Z = \frac{X}{Y^{\frac{1}{q}}}$  has a BSGHN distribution.

Consider the transformation  $w = y^{\frac{1}{q}}$  with corresponding Jacobian  $\mathcal{J}(x, y \rightarrow z, w) = qw^q$ , then

$$f_{Z,W}(z, w) = f_{X,Y}(zw, w^q) \mathcal{J} = qw^q f_X(zw) f_Y(w^q)$$

with  $X$  and  $Y$  independent.

Using Taylor’s series expansion of the exponential function, we obtain

$$\begin{aligned} f_Z(z) &= \int_0^1 f_{Z,W}(z, w) dw \\ &= \frac{2\alpha q z^{\alpha-1}}{\sqrt{2\pi} \sigma^\alpha B(a, b)} \int_0^1 w^{\alpha+aq-1} (1-w^q)^{b-1} \exp\left[-\frac{1}{2\sigma^2\alpha} (zw)^{2\alpha}\right] dw \\ &= \frac{2\alpha q z^{\alpha-1}}{\sqrt{2\pi} \sigma^\alpha B(a, b)} \sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(i+1)} \left(\frac{z^{2\alpha}}{2\sigma^2\alpha}\right)^i \int_0^1 w^{2\alpha i + \alpha + aq - 1} (1-w^q)^{b-1} dw \\ &= \frac{2\alpha q z^{\alpha-1}}{\sqrt{2\pi} \sigma^\alpha B(a, b)} \sum_{i=0}^{\infty} \frac{(-1)^i}{\Gamma(i+1)} \left(\frac{z^{2\alpha}}{2\sigma^2\alpha}\right)^i \frac{1}{q} B(2\alpha i + \alpha + aq - q + 1, b) \end{aligned} \tag{1}$$

which is the density of the BSGHN distribution. In a similar fashion, it can be shown that the r.v.  $R = Y^{-\frac{1}{q}}$  has the density function

$$f_R(r) = \frac{1}{qB(a, b)} r^{\frac{a}{q}-1} \left(1 - r^{\frac{1}{q}}\right)^{b-1}, \quad 0 < r < 1.$$

The proof follows by noting that  $Z = XY^{-\frac{1}{q}} = XR$ . ■

Let  $G(\cdot)$  be a cdf, Eugene, Lee and Famoye (2002) defined a class of generalised distributions from  $G(\cdot)$  as

$$F_X(x) = \frac{1}{B(a, b)} \int_0^{G(x)} u^{a-1} (1-u)^{b-1} du, \quad 0 < i < 1 \tag{2}$$

where the shape parameters  $a, b \in \mathbb{R}^+$  control the skewness and tail weight of the distribution. In other words,  $X = G^{-1}(Y)$  has a cdf as in (2) if  $Y \sim B(a, b)$ . Many researchers generalise distributions by replacing  $G(\cdot)$  in (2) by a cdf of a lifetime distribution. The reader is referred to Famoye, Lee and Olumolade (2005) and Barreto-Souza, Santos and Cordeiro (2010), to mention a few.

All the parameters, except the scale parameter  $\sigma$ , are shape-type parameters. Specifically, with reference to Theorem 4, the shape parameter  $q$  is the reciprocal-shape that gives specific distributions for limiting cases. The shape parameter  $\alpha$  is a tuning parameter which controls the half-normal component of the density function. The shape parameters  $a$  and  $b$  are beta-component parameters.

**Remark 1** The distribution of  $R$  given in Theorem 1 is the generalised beta distribution proposed by McDonald (1984) which can be obtained by taking  $G(\cdot)$  in (2) to be the cdf of a  $B(\frac{1}{q}, 1)$ . Two important special cases of this generalised beta distribution are the beta distribution if  $q$  is equal to 1, and the Kumaraswamy distribution if  $a$  is equal to 1. This follows from (2) since

$$f_X(x) = \frac{1}{B(a, b)} g(x) [G(x)]^{a-1} [1 - G(x)]^{b-1},$$

with  $g(\cdot) = \frac{dG(x)}{dx}$ . For  $B(\frac{1}{q}, 1)$  we have  $g(x) = \frac{1}{q} x^{\frac{1}{q}-1}$  and  $G(x) = x^{\frac{1}{q}}$ .

The following theorem present a hypergeometric representation of the BSGHN distribution for a restricted case.

**Theorem 2** Let  $Z \sim BSGHN(\sigma, \alpha, q, a, b)$  and assume that  $n = \frac{2\alpha}{q} \geq 2$  is an integer number. The density function of  $Z$  can be expressed by the following hypergeometric form

$$f_Z(z) = \sqrt{\frac{2}{\pi}} \frac{\alpha B(b, \nu)}{\sigma^\alpha B(a, b)} z^{\alpha-1} {}_nF_n \left( \lambda_1; \lambda_2; -\frac{1}{2} \left( \frac{z}{\sigma} \right)^{2\alpha} \right),$$

where  $\nu = a + \frac{n}{2}$ ,  ${}_nF_n(\cdot; \cdot; \cdot)$  the confluent hypergeometric function and

$$\lambda_1 = \left( \frac{\nu}{n}, \frac{\nu+1}{n}, \dots, \frac{\nu+n-1}{n} \right), \lambda_2 = \left( \frac{b+\nu}{n}, \frac{b+\nu+1}{n}, \dots, \frac{b+\nu+n-1}{n} \right).$$

**Proof.** Consider equation (1) in Theorem 1 and the transformation  $y = w^q$  with corresponding Jacobian  $\mathcal{J}(w \rightarrow y) = \frac{1}{q} y^{-\frac{1}{q}(q-1)}$ , then

$$f_Z(z) = \frac{2\alpha z^{\alpha-1}}{\sqrt{2\pi} \sigma^\alpha B(a, b)} \int_0^1 y^{\alpha+\frac{\alpha}{q}-1} (1-y)^{b-1} \exp \left[ -\frac{1}{2} \left( \frac{z}{\sigma} \right)^{2\alpha} u^{\frac{2\alpha}{q}} \right] dy.$$

The results follows from equation 3 on page 370 of Gradshteyn and Ryzhik (2007). ■

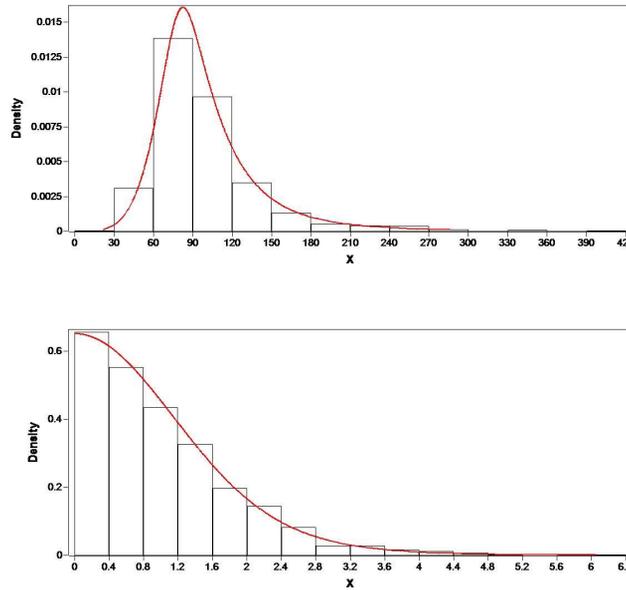
To generate a set of random numbers from the BSGHN distribution we follow the procedure as in Olmos et al. (2014), with a minor change based on the result from Theorem 1. The generation procedure is given by the algorithm below.

**Algorithm A** Generating random values from the  $Z \sim BSGHN(\sigma, \alpha, q, a, b)$  distribution

- (1) Generate a random number,  $N$ , from a  $N(0, 1)$ .
- (2) Compute  $X = \sigma |N|^{\frac{1}{\alpha}}$ .
- (3) Generate a random number,  $Y$ , from a  $B(a, b)$ .
- (4) Compute  $W = Y^{\frac{1}{q}}$ .
- (5) Obtain the random number  $Z = \frac{X}{W}$ .

Note that the algorithm can also be defined in terms of  $R$  by generating a random number from the distribution given by Theorem 1.

Figure 3 represents random numbers generated using Algorithm A. A thousand numbers were generated from the BSGHN distribution for the unimodal case corresponding with the red unimodal distribution of Figure 2. A thousand numbers were also generated for the corresponding red truncated distribution from Figure 2.



**Figure 3:** Histograms of random numbers from the *BSGHN* distribution corresponding to the *red* distributions from Figure 2.

### 3. Some Characteristics of the BSGHN

Theorem 3 present the characteristic function (cf) and cdf's of the BSGHN distribution. The *k*th moment about the origin, skewness, kurtosis, mean and median deviation of the BSGHN distribution is given in Theorem 4.

**Theorem 3** Let  $Z \sim BSGHN(\sigma, \alpha, q, a, b)$ .

1. The cf of *Z* is given by

$$\psi_Z(t) = \frac{2\alpha}{\sqrt{2\pi}\sigma^\alpha B(a, b)} \sum_{j=0}^{\infty} \left(\frac{1}{2\sigma^{2\alpha}}\right)^j \frac{B(\alpha(2j+1) + q(a-1) + 1, b)}{\Gamma(j+1)} \cdot \frac{(-1)^{j(2\alpha+1)+1} \Gamma(\alpha(2j+1))}{(it)^{\alpha(2j+1)}}.$$

2. The cdf of *Z* is given by

$$F(x) = P(Z \leq x) = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma^\alpha B(a, b)} \sum_{i=0}^{\infty} \left(\frac{-1}{2\sigma^{2\alpha}}\right)^i \frac{B(c_i, b)}{(2i+1)\Gamma(i+1)} x^{\alpha(2i+1)},$$

with  $c_i$  as defined in Definition 3.

**Proof.** From Definition 3 we have

$$\begin{aligned} \psi_Z(t) &= E(e^{itZ}) \\ &= \frac{2\alpha}{\sqrt{2\pi}\sigma^\alpha B(a,b)} \sum_{j=0}^{\infty} \left(\frac{-1}{2\sigma^{2\alpha}}\right)^j \frac{B(\alpha(2j+1)+q(a-1)+1,b)}{\Gamma(j+1)} \\ &\quad \cdot \int_0^{\infty} z^{\alpha(2j+1)-1} e^{itz} dz \\ &= \frac{2\alpha}{\sqrt{2\pi}\sigma^\alpha B(a,b)} \sum_{j=0}^{\infty} \left(\frac{1}{2\sigma^{2\alpha}}\right)^j \frac{B(\alpha(2j+1)+q(a-1)+1,b)}{\Gamma(j+1)} \\ &\quad \cdot \frac{(-1)^{j(2\alpha+1)+1} \Gamma(\alpha(2j+1))}{(it)^{\alpha(2j+1)}}. \end{aligned}$$

2. The cdf follows from integrating  $f_Z$  as defined by Definition 3. ■

**Theorem 4** Let  $Z \sim BSGHN(\sigma, \alpha, q, a, b)$ .

1. The  $k$ th moment about the origin of  $Z$  is

$$\mu_k = E(Z^k) = \eta_k \sigma^k \quad \text{with} \quad \eta_k = \sqrt{\frac{2^{\frac{k}{\alpha}}}{\pi}} \Gamma\left(\frac{k+\alpha}{2\alpha}\right) \frac{B(qk, a)}{B(a, b)}.$$

2. Skewness and kurtosis of  $Z$  are given by

$$\gamma_1 = \frac{\eta_3 - 3\eta_1\eta_2 + 2\eta_1^3}{(\eta_2 - \eta_1^2)^{\frac{3}{2}}} \quad \text{and} \quad \gamma_2 = \frac{\eta_4 - 4\eta_1\eta_3 + 6\eta_1^2\eta_2 - 3\eta_1^4}{(\eta_2 - \eta_1^2)^2} - 3$$

respectively.

3. The mean deviation and median deviation are given by

$$\delta_1(Z) = 2\mu_1 F(\mu_1) - 2\mu_1 + 2I(\mu_1) \quad \text{and} \quad \delta_2(Z) = 2I(M) - \mu_1$$

respectively, with

$$I(c) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\sigma^\alpha B(a,b)} \sum_{i=0}^{\infty} \left(\frac{-1}{2\sigma^{2\alpha}}\right)^i \frac{B(c_i, b)}{[\alpha(2i+1)+1]\Gamma(i+1)} c^{\alpha(2i+1)+1}. \tag{3}$$

**Proof.**

1. Since  $R$  and  $X$  are independent, using Theorem 1 and the result from Cooray and Ananda (2008) the proof follows.

3. The mean deviation and median deviation are respectively defined as

$$\delta_1(Z) = \int_0^{\infty} |z - \mu_1| f(z) dz \quad \text{and} \quad \delta_2(Z) = \int_0^{\infty} |z - M| f(z) dz$$

with  $M$  the median of the distribution of  $Z$ .

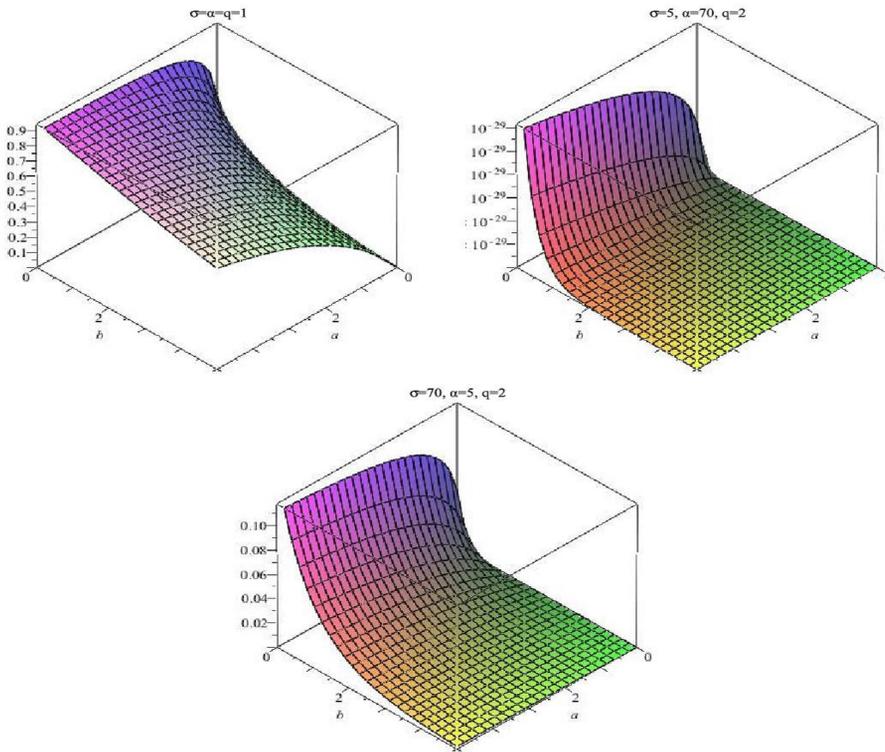
Let  $I(c) = \int_0^c zf(z)dz$ . The measures  $\delta_1(Z)$  and  $\delta_2(Z)$  can then be written as

$$\delta_1(Z) = 2\mu_1F(\mu_1) - 2\mu_1 + 2I(\mu_1) \text{ and } \delta_2(Z) = 2I(M) - \mu_1.$$

Expression (3) follows from Definition 3. ■

Graphs of the cdf of the BSGHN distribution are displayed in Figure 3 for different parameter values. These graphs show the sensitivity of the distribution with respect to the parameters of the beta component in Theorem 1. As the parameters  $a$  and  $b$  get larger, the behaviour substantially changes.

We experienced difficulties drawing the cdf for large values of the shape parameters  $a$  and  $b$ . To be more specific, we can determine relevant choices of the parameters  $a$  and  $b$ , by considering the behaviour of skewness and kurtosis characteristics.



**Figure 4:** Cdf's of the BSGHN distribution for selected parameter values.

Skewness and kurtosis of the BSGHN distribution are tabulated in Table 1 for different parameter values. It can be seen that the characteristics perform worse when the beta component parameters, as in Definition 3, increase. However there is no clear monotonic trend based on these parameters. The trend changes somewhat when we focus on the behaviour of  $\alpha$  and  $q$  for fixed parameter values  $a$  and  $b$ . In this respect, for fixed  $\alpha$ , the skewness and kurtosis increase as  $q$  increases and for fixed  $q$ , the skewness and kurtosis decrease as  $\alpha$  increases. In conclusion, care should be taken in selecting the

beta component parameters  $a$  and  $b$ . Since the beta distribution simplifies to the uniform distribution by taking  $a = b = 1$ , we call the selection  $a = b = 1$  the safe mode. The safe mode can be used for selecting initial values during numerical iteration procedures. Care should be taken when the safe mode is not used. Cases, not using the safe mode, fits data better as shown later by fitting the distribution to a real data set.

The mean deviation and median deviation can be used to derive directly the Bonferroni and Lorenz curves with applications in economics, reliability, demography, insurance and medicine.

**Table 1:** Skewness,  $\gamma_1$  and Kurtosis,  $\gamma_2$  of the BSGHN distribution.

$\alpha$	$q$	$a$	$b = 0.1$		$b = 1$		$b = 3$		$b = 5$	
			$\gamma_1$	$\gamma_2$	$\gamma_1$	$\gamma_2$	$\gamma_1$	$\gamma_2$	$\gamma_1$	$\gamma_2$
3	2	0.1	0.5478	-1.2080	0.8955	-4.1522	15.9427	-95.7950	6.3813	-27.7447
3	2	1	4.4216	22.0480	15.0376	-18.4684	2.1599	-6.5906	2.0439	-6.1676
3	2	3	9.0146	111.2584	18.2577	-19.6467	2.0099	-6.0380	2.0007	-6.0029
3	2	5	11.8705	211.9780	6.0084	-9.9020	2.0017	-6.0066	2.0000	-6.0001
3	5	0.1	0.7040	-1.0408	-0.2798	-0.8168	2.2231	-10.0149	56.4728	-558.3012
3	5	1	7.2345	60.6559	1.9199	3.4332	16.2364	-57.9381	2.5626	-7.9853
3	5	3	27.1696	1055.2549	5.5575	44.0277	3.9662	-9.8083	2.0556	-6.1979
3	5	5	54.7457	4987.3845	9.1500	138.5434	2.7891	-7.3918	2.0112	-6.0416
3	10	0.1	0.8152	-0.8807	-0.2690	-0.9829	-0.0830	-1.0948	1.0191	-4.7150
3	10	1	10.3483	125.1485	2.9567	9.3481	1.6616	2.0639	15.0376	-18.4654
3	10	3	68.9504	6928.1077	13.6148	270.2411	4.9850	34.6947	444.6275	3231.3789
3	10	5	223.5355	87769.0328	35.1776	2172.1601	9.0951	138.3562	23.5318	117.9247
5	2	0.1	0.3332	-1.5904	2.2014	-17.1118	5.7505	-26.1421	3.9999	-15.5533
5	2	1	4.1299	18.4879	50.2689	-189.8781	2.1292	-6.4857	2.0362	-6.1397
5	2	3	8.4888	96.2428	11.0643	-20.3584	2.0084	-6.0325	2.0006	-6.0025
5	2	5	11.1920	184.0777	4.9342	-10.3525	2.0014	-6.0057	2.0000	-6.0001
5	5	0.1	0.5023	-1.4752	-0.7964	-0.3787	15.3254	-144.9026	13.7064	-91.1874
5	5	1	6.8027	52.1133	1.7139	2.2980	8.6614	-31.0225	2.4349	-7.5841
5	5	3	25.6440	918.5673	5.2202	37.8166	3.5169	-9.4371	2.0471	-6.1703
5	5	5	51.6836	4344.0643	8.6388	120.6629	2.6493	-7.3324	2.0097	-6.0361
5	10	0.1	0.6196	-1.3526	-0.7040	-0.7568	-0.5447	-1.1626	3.4152	-26.8083
5	10	1	9.7506	108.2889	2.7256	7.4336	1.4690	1.0825	50.2689	-189.8781
5	10	3	65.0934	6034.2112	12.8414	234.8284	4.7056	29.9444	64.0467	121.8831
5	10	5	211.0394	76452.2076	33.2122	1892.2631	8.6374	121.6026	15.9131	48.1847

In the context of reliability, the stress-strength model describes a system whereby a strength  $Z_1$  is subjected to a stress  $Z_2$ . In terms of the lifetime of a component, the component fails in instants where the stress applied exceeds the strength level, that is  $Z_1 > Z_2$ . Hence,  $R = P(Z_2 < Z_1)$  is a measure of component life time reliability. The stress-strength model has many applications, for example in engineering applications such as the deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft and the ageing of concrete pressure vessels. In the context of biometrical studies, the random variable  $Z_1$  may represent the lifetime of a patient treated with a certain drug while  $Z_2$  represents the lifetime when treated with a placebo.

The theorem below give the reliability,  $R$ , for the case where  $Z_1$  and  $Z_2$  are independently distributed BSGHN random variables.

**Theorem 5** Let  $Z_1$  and  $Z_2$  be independent, with  $Z_1 \sim BSGHN(\sigma_1, \alpha_1, q_1, a_1, b_1)$  and  $Z_2 \sim BSGHN(\sigma_2, \alpha_2, q_2, a_2, b_2)$ . The reliability is then

$$R = \frac{\sqrt{2}q_1}{\pi\sigma_2^{\alpha_2}B(a_1, b_1)B(a_2, b_2)} \cdot \sum_{i=0}^{\infty} \frac{B(c_i^{(2)}, b_2) \Gamma\left(\frac{\alpha_1 + \alpha_2(2i+1)}{2\alpha_1}\right) \left(\frac{-1}{2\sigma_2^{2\alpha_2}}\right)^i (2\sigma_1^{2\alpha_1})^{\frac{\alpha_2(2i+1)}{2\alpha_1}}}{(2i+1)\Gamma(i+1)} \cdot \int_0^1 w^{a_1q_1-1-\alpha_2(2i+1)} (1-w^{q_1})^{b_1-1} dw$$

**Proof.** Let  $c_k^{(s)} = \alpha_s(2k+1) + q_s(a_s - 1) + 1$ . The proof follows from (1) and the cdf as given in Theorem 3 since

$$\begin{aligned} R &= \int_0^\infty f_{Z_1}(z)F_{Z_2}(z) dz \\ &= \int_0^1 \frac{2\alpha_1q_1w^{\alpha_1+a_1q_1-1}(1-w^{q_1})^{b_1-1}}{\pi\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}B(a_1, b_1)B(a_2, b_2)} \sum_{i=0}^{\infty} \left(\frac{-1}{2\sigma_2^{2\alpha_2}}\right)^i \frac{B(c_i^{(2)}, b_2)}{(2i+1)\Gamma(i+1)} \\ &\quad \cdot \int_0^\infty z^{\alpha_1-1+\alpha_2(2i+1)} \exp\left[-\frac{1}{2\sigma_1^{2\alpha_1}}(zw)^{2\alpha_1}\right] dz dw \\ &= \frac{2\alpha_1q_1}{\pi\sigma_1^{\alpha_1}\sigma_2^{\alpha_2}B(a_1, b_1)B(a_2, b_2)} \int_0^1 \sum_{i=0}^{\infty} \left(\frac{-1}{2\sigma_2^{2\alpha_2}}\right)^i \frac{B(c_i^{(2)}, b_2)}{(2i+1)\Gamma(i+1)} \\ &\quad \cdot w^{\alpha_1+a_1q_1-1}(1-w^{q_1})^{b_1-1} \frac{\Gamma\left(\frac{\alpha_1 + \alpha_2(2i+1)}{2\alpha_1}\right)}{2\alpha_1} \left(\frac{w^{2\alpha_1}}{2\sigma_1^{2\alpha_1}}\right)^{\frac{\alpha_1 + \alpha_2(2i+1)}{2\alpha_1}} dw \end{aligned}$$

from which the result follows. ■

As a numerical example to the reliability result from Theorem 5 we consider unimodal distributions as presented in Figure 2. Using numerical integration and selecting the strength  $Z_1 \sim BSGHN(70, 5, 2, 2, 2)$ , red distribution in Figure 2, and  $Z_2 \sim BSGHN(70, 5, 2, 0.5, b)$  a reliability of 0.81 is obtained if  $b = 2.0$ . Table 3 presents reliability values, changing the value of  $b$  from 0.5 to 2.0, reaching the black distribution, in Figure 2, when  $b = 2.0$ . As expected, the reliability tends to  $\frac{1}{2}$  as the distributions become more similar.

The truncated case is also considered, selecting the strength,  $Z_1 \sim BSGHN(1, 1, 4, 1.1, 1)$ . Table 2 considers reliability values for  $Z_2 \sim BSGHN(1, 1, 4, a, b)$ , moving from the blue distribution in Figure 2, to the green distribution and then to the black distribution. Once again, the reliability tend to  $\frac{1}{2}$ .

### 4. Estimation

In this section, we obtain estimates for the parameters through an implementation of the EM algorithm, adopting the procedure in Moradi, Arashi, Arslan and Iranmanesh (2017). Let  $\Theta = (\sigma, \alpha, q, a, b) \in$

**Table 2:** Reliability values for selected distributions from Figure 2.

Unimodal, $Z_1 \sim BSGHN(70, 5, 2, 2, 2)$ and $Z_2 \sim BSGHN(70, 5, 2, 0.5, b)$											
$b$	2.00	1.75	1.50	1.25	1.00	0.75	0.50				
$R$	0.81	0.79	0.76	0.73	0.68	0.62	0.53				
Truncated, $Z_1 \sim BSGHN(1, 1, 4, 1.1, 1)$ and $Z_2 \sim BSGHN(1, 1, 4, a, b)$											
$a$	0.10	0.20	0.30	0.40	0.50	0.05	0.05	0.05	0.05	0.05	0.05
$b$	2.00	2.00	2.00	2.00	2.00	1.75	1.50	1.25	1.00	0.75	0.50
$R$	0.83	0.74	0.68	0.65	0.62	0.61	0.60	0.59	0.57	0.55	0.53

$\mathbb{R}^{5+}$  denote the parameter space and  $Z \sim BHN(\sigma, \alpha, q, a, b)$ . Assume that  $Z_1, \dots, Z_n$  are independently distributed and identical to the distribution of  $Z$ . By making use of equation (1), the log-likelihood function can be written as

$$l(\Theta) = 0.35n + n \ln(\alpha) + n \ln(q) - n\alpha \ln(\sigma) - n \ln[B(a, b)] + (\alpha - 1) \sum_{i=1}^n \ln(z_i) + \sum_{i=1}^n \ln \left\{ \int_0^1 w^{\alpha+aq-1} (1-w^q)^{b-1} \exp \left[ -\frac{1}{2\sigma^{2\alpha}} (z_i w)^{2\alpha} \right] dw \right\}. \tag{4}$$

Suppose  $W_1, \dots, W_n$  are mixing latent variables and are observable. The complete data set is then given by  $(Z_i, W_i), i = 1, \dots, n$ . The joint density of  $(Z_i, W_i)$  is

$$f(z_i, w_i) = \frac{2\alpha q z_i^{\alpha-1}}{\sqrt{2\pi\sigma^{2\alpha}} B(a, b)} w_i^{\alpha+aq-1} (1-w_i^q)^{b-1} \exp \left[ -\frac{1}{2\sigma^{2\alpha}} (z_i w_i)^{2\alpha} \right].$$

Thus accordance with (4), the complete log-likelihood is

$$l_c(\Theta) = 0.35n + n \ln(\alpha) + n \ln(q) - n\alpha \ln(\sigma) - n \ln[B(a, b)] + (\alpha - 1) \sum_{i=1}^n \ln(z_i) + (\alpha + aq - 1) \sum_{i=1}^n \ln(w_i) + (b - 1) \sum_{i=1}^n \ln(1 - w_i^q) - \sum_{i=1}^n \frac{1}{2\sigma^{2\alpha}} (z_i w_i)^{2\alpha}.$$

In reality, the latent variables  $W_i, i = 1, \dots, n$  are not observable. As a remedy, we consider the conditional expectation of  $l_c(\Theta)$  given the observations  $z_i$  and the estimated parameters  $(\hat{\Theta})$ . For this we have

$$Q(\Theta) = E(l_c(\Theta) | z_i, \hat{\Theta}) = 0.35n + n \ln(\alpha) + n \ln(q) - n\alpha \ln(\sigma) - n \ln[B(a, b)] + (\alpha - 1) \sum_{i=1}^n \ln(z_i) + (\alpha + aq - 1) \sum_{i=1}^n E(\ln(W_i) | z_i, \hat{\Theta}) + (b - 1) \sum_{i=1}^n E(\ln(1 - W_i^q) | z_i, \hat{\Theta}) - \sum_{i=1}^n \frac{1}{2\sigma^{2\alpha}} z_i^{2\alpha} E(W_i^{2\alpha} | z_i, \hat{\Theta}). \tag{5}$$

Using the conditional distribution of  $W$  given  $Z$ , we obtain

$$E_i^{(1)} = E(\ln(W)|z_i, \hat{\Theta}) = \int_0^1 \frac{\ln(w)w^{\hat{\alpha}+\hat{a}\hat{q}-1}(1-w^{\hat{q}})^{\hat{b}-1} \exp\left[-\frac{1}{2\hat{\sigma}^{2\hat{\alpha}}}(z_i w)^{2\hat{\alpha}}\right]}{\int_0^1 w^{\hat{\alpha}+\hat{a}\hat{q}-1}(1-w^{\hat{q}})^{\hat{b}-1} \exp\left[-\frac{1}{2\hat{\sigma}^{2\hat{\alpha}}}(z_i w)^{2\hat{\alpha}}\right]} dw,$$

$$E_i^{(2)} = E(\ln(1-W^q)|z_i, \hat{\Theta}) = \int_0^1 \frac{\ln(1-w^{\hat{q}})w^{\hat{\alpha}+\hat{a}\hat{q}-1}(1-w^{\hat{q}})^{\hat{b}-1} \exp\left[-\frac{1}{2\hat{\sigma}^{2\hat{\alpha}}}(z_i w)^{2\hat{\alpha}}\right]}{\int_0^1 w^{\hat{\alpha}+\hat{a}\hat{q}-1}(1-w^{\hat{q}})^{\hat{b}-1} \exp\left[-\frac{1}{2\hat{\sigma}^{2\hat{\alpha}}}(z_i w)^{2\hat{\alpha}}\right]} dw$$

and

$$E_i^{(3)} = E(W_i^{2\alpha}|z_i, \hat{\Theta}) = \int_0^1 \frac{w^{3\hat{\alpha}+\hat{a}\hat{q}-1}(1-w^{\hat{q}})^{\hat{b}-1} \exp\left[-\frac{1}{2\hat{\sigma}^{2\hat{\alpha}}}(z_i w)^{2\hat{\alpha}}\right]}{\int_0^1 w^{\hat{\alpha}+\hat{a}\hat{q}-1}(1-w^{\hat{q}})^{\hat{b}-1} \exp\left[-\frac{1}{2\hat{\sigma}^{2\hat{\alpha}}}(z_i w)^{2\hat{\alpha}}\right]} dw.$$

Substituting  $E_i^{(i)}$ ,  $i = 1, 2, 3$  in to (5) we obtain

$$Q(\Theta) = 0.35n + n\ln(\alpha) + n\ln(q) - n\alpha\ln(\sigma) - n\ln[B(a, b)] + (\alpha - 1) \sum_{i=1}^n \ln(z_i) + (\alpha + aq - 1) \sum_{i=1}^n E_i^{(1)} + (b - 1) \sum_{i=1}^n E_i^{(2)} - \sum_{i=1}^n \frac{1}{2\sigma^{2\alpha}} z_i^{2\alpha} E_i^{(3)}. \tag{6}$$

Taking derivatives w.r.t the parameter  $q$ ,  $a$  and  $b$  and setting the expressions equal to zero to obtain the system of equations

$$\begin{cases} \frac{\partial Q(\Theta)}{\partial q} = \frac{1}{q} + a\bar{E}^{(1)} = 0 \\ \frac{\partial Q(\Theta)}{\partial a} = -\psi(a) + \psi(a+b) + q\bar{E}^{(1)} = 0 \\ \frac{\partial Q(\Theta)}{\partial b} = -\psi(b) + \psi(a+b) + \bar{E}^{(2)} = 0 \end{cases}$$

where  $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$  and  $\bar{E}^{(j)} = \frac{1}{n} \sum_{i=1}^n E_i^{(j)}$ ,  $j = 1, 2$ .

To solve the system of equations, we fix the value of the parameter  $a$  and then find the solution of  $b$  numerically. It is also possible to fix the value of the parameter  $b$  and then find  $a$  numerically. Using these estimated values,  $\hat{a}$  and  $\hat{b}$  a solution for  $q$  can be obtained. Substituting the estimates of  $a$ ,  $b$  and  $q$  into (6),  $Q(\Theta)$  can be maximize w.r.t the parameters  $\alpha$  and  $\sigma$ . Maximizing  $Q(\Theta)$  for  $\sigma$  and  $\alpha$ , the equations

$$\hat{\sigma} = \sqrt[2\hat{\alpha}]{\frac{1}{n} \sum_{i=1}^n z_i^{2\hat{\alpha}} E_i^{(3)}} \tag{7}$$

and

$$\frac{1}{\hat{\alpha}} = \ln(\hat{\sigma}) - \frac{1}{n} \sum_{i=1}^n \ln(z_i) - \bar{E}^{(1)} + \frac{1}{2n} \sum_{i=1}^n \left(\frac{z_i}{\hat{\sigma}^2}\right)^{\hat{\alpha}} \ln\left(\frac{z_i}{\hat{\sigma}^2}\right) E_i^{(3)}$$

are obtained. Substituting the expression for  $\hat{\sigma}$  into the equation above, yields after some algebra

$$\frac{1}{2} \ln(\zeta_{\hat{\alpha}}) \left( 1 - \frac{1}{n\zeta_{\hat{\alpha}}} \sum_{i=1}^n z_i^2 E_i^{(3)} \right) - \frac{\hat{\alpha}}{n} \sum_{i=1}^n \ln(z_i) - \hat{\alpha} \bar{E}^{(1)} - 1 = 0 \tag{8}$$

where  $\zeta_{\hat{\alpha}} = \frac{1}{n} \sum_{i=1}^n z_i^2 \hat{\alpha} E_i^{(3)}$ . Using (8) a solution for  $\hat{\alpha}$  can numerically be obtained where after a solution for  $\hat{\sigma}$  can be calculated form (7).

The procedure above was coded using the PROC IML procedure of the SAS system. The numerical integration subroutine *QUAD* was used to evaluate  $E_i^{(i)}$ ,  $i = 1, 2, 3$  numerically after determining the normalization constant also using the *QUAD* subroutine. Estimates were firstly obtained for  $a$ ,  $b$  and  $q$ . Solutions were then obtained for  $\sigma$  and  $\alpha$  using the values of  $E_i^{(i)}$ ,  $i = 1, 2, 3$  as determined during the last iteration in obtaining  $a$ ,  $b$  and  $q$ .

### 5. Simulation Study

Table 3 presents the absolute bias results obtained from a simulation study following the estimation procedure as in paragraph 4. Random numbers are generated from a truncated *BSGHN*(1, 1, 4, 1, 1) distribution and a unimodal *BSGHN*(1600, 3, 12, 3, 1) distribution respectively for sample sizes ranging between 30 and 150. Five hundred simulation repetitions were performed. (The unimodal distribution parameters correspond to the parameter estimates as in (6).) The results show that, as the sample size increases, the bias tend to decrease with the largest effect for the parameters  $q$  and  $\sigma$  when compared to other parameters. It shows that the estimation bias is sensitive to sample size.

**Table 3:** Absolute bias for a truncated and unimodal *BSGHN*.

Truncated					Unimodal				
$n$	$b$	$q$	$\alpha$	$\sigma$	$n$	$b$	$q$	$\alpha$	$\sigma$
30	0.0017	0.8569	0.1706	0.1294	30	0.0030	1.6104	0.3003	58.1551
50	0.0018	0.7024	0.1703	0.1025	50	0.0025	1.2968	0.2504	43.1193
80	0.0018	0.5373	0.1660	0.0815	80	0.0020	0.9790	0.2125	35.0508
100	0.0019	0.4931	0.1758	0.0776	100	0.0017	0.8110	0.2210	31.2119
120	0.0019	0.4598	0.1693	0.0672	120	0.0015	0.7254	0.2091	30.1841
150	0.0020	0.4434	0.1703	0.0624	150	0.0013	0.6148	0.2019	26.8372

### 6. Application

In this section, we fit the proposed model to the same data set as been considered by Olmos et al. (2014) and compare the fitted models by calculating the likelihood and model selection criteria: Akaike information criterion (*AIC*), second order Akaike information criterion (*AICc*) and Bayesian information criterion (*BIC*). The three criteria are defined, for sample size  $n$  and  $k$  free parameters in the model, as

$$AIC = 2k - 2\ln(\text{likelihood}), \quad AICc = AIC + \frac{2k(k+1)}{n-k-1},$$

and

$$BIC = k \ln(n) - 2 \ln(\text{likelihood}).$$

The preferred model is the model with the smallest values for the *AIC*, *BIC* and *AICc*. In the case of a small sample size or large number of parameters, the *AICc* measure is preferred over the *AIC* measure.

The data set considered by Olmos et al. (2014) consists of 101 observed lifetimes in cycles  $10^{-3}$  of aluminium 6061-T6 pieces cut in parallel with the rolling direction, oscillating at 18 Hz at a maximum pressure of 21000 psi. The results of fitting different models to the data are reported in Table 4. The BSGHN model is fitted following the estimation procedure as in paragraph 4 keeping  $a$  constant at 3.2. It is apparent that the BSGHN model gives a better fit since the values of the fitting criteria are smaller.

**Table 4:** Fitting statistics for the data from Olmos et al. (2014).

Parameter	GHN	SGHN	BSGHN
$\sigma$	1629.278 (42.251)	1452.068 (76.575)	1589.053 (38.636)
$\alpha$	2.996 (0.235)	3.480 (0.409)	2.960 (0.199)
$q$	-	10.220 (4.183)	11.540 (1.068)
$b$	-	-	1.010 (0.002)
Log-likelihood	-748.529	-747.016	-690.288
AIC	1501.058	1500.032	1390.578
BIC	1506.288	1507.877	1394.423
AICc	1501.180	1500.279	1391.209

Standard deviations are given within brackets

## 7. Conclusions

In this paper a new distribution, the BSGHN distribution, is introduced by extending the work of Olmos et al. (2014). This new distribution is obtained by replacing the uniform distribution in the GHN distribution with the beta distribution. The proposed distribution has more flexibility than the latter due to the additional shape parameters which is a result of the structure of the beta component.

## Acknowledgements

The authors would like to thank Ms Norouzirad, University of Shahrood, for her assistance. This work was based on research supported by STATOMET, Department of Statistics, University of P-retoria.

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