South African Statistical Journal
2018, Vol. 52, No. 2, 115–128

MAXIMUM LIKELIHOOD ESTIMATION OF THE GENERALISED GOMPERTZ DISTRIBUTION UNDER PROGRESSIVELY FIRST-FAILURE CENSORED SAMPLING

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In this paper, the maximum likelihood estimators of the unknown parameters, as well as some lifetime parameters survival and hazard rate functions, of a three-parameter generalised Gompertz lifetime model based on progressively first-failure censored sampling are obtained. Approximate confidence intervals for the unknown parameters and the reliability characteristics are constructed based on the s-normal approximation to the asymptotic distribution of maximum likelihood estimators. Although the proposed estimators cannot be expressed in explicit forms, these can be easily obtained through the use of appropriate numerical techniques. Finally, a real data set has been analysed for illustrative purposes.

Key words: Exponentiated Gompertz distribution, Maximum likelihood estimation, Progressively first-failure censored sampling, Reliability characteristics.

1. Introduction

In life testing and reliability studies, the experimenter may not always obtain complete information on failure times for all experimental units. Data obtained from such experiments are called censored data. Cohen (1963) suggested Type-I and Type-II progressive censored samples (PCSs) to overcome the drawback of conventional Type-I and Type-II censoring schemes. The PCSs enables an efficient exploitation of the available resources by continual removal of a pre-specified number of surviving test units at each failure time. On the other hand, the removal of units before failure may be intentional to save time and cost or when some items have to be removed for use in another experiment, for more detail, see Balakrishnan and Cramer (2014). The Gompertz distribution (GD) was originally introduced by Gompertz (1825) and is one of the classical mathematical models that represent the survival function based on laws of mortality. This lifetime model plays an important role in modelling human mortality as well as fitting actuarial tables. The three-parameter generalised Gompertz distribution (GGD) with the bathtub shape or increasing failure rate function was suggested by El-Gohary, Alshamrani and Al-Otaibi (2013), as was done for the exponentiated Weibull distribution

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MSC2010 subject classifications. 62F10, 62N01, 62N02.

by Mudholkar, Srivastava and Freimer (1995), to provide a better fit for the real data than very well-known distributions such as the exponential distribution (ED), the Gompertz distribution (GD) and the generalised exponential distribution (GED). The GGD contains some models as special cases: if putting $\theta = 0$, the two-parameter GED(α, λ) can be derived; if putting $\alpha = 1$, the two-parameter GD(λ, θ) can be derived; also, the one-parameter ED(λ) can be derived by setting $\theta = 0$ and $\alpha = 1$. Further, the hazard rate function of the GGD is a decreasing function with $\theta = 0$ and $\alpha < 1$, it is an increasing function for $\alpha > 1$, and it is a bathtub for $\theta > 0$ and $\alpha < 1$. For more detail, see El-Gohary et al. (2013).

Recently, many statistical inferences of the unknown parameters of some reliability models based on progressive first-failure censored sampling (PFFCS) and Type-II PCS have been studied by several authors, such as: Soliman, Abd-Ellah, Abou-Elheggag and Abd-Elmougod (2012) obtained the maximum likelihood and Bayes estimators of the parameters of GD based on PFFCS, Soliman and Al Sobhi (2015) discussed the maximum likelihood estimators (MLEs) and Bayes estimates of GD under PFFCS, Ahmed (2015) studied the maximum likelihood and Bayesian estimations of parameters of a GGD based on Type-II PCS, Demġr and Saraçoğlu (2015) obtained the MLEs for the parameters of the GGD under Type-II PCS, also, Abu-Zinadah and Bakoban (2017) discussed the MLEs and Bayes estimators for two shape parameters and reliability characteristic of GGD based on Type-II PCS.

The main aim of this paper is to discuss the likelihood inference of the unknown parameters, survival function (SF) and hazard rate function (HRF) of GGD compared with ED and GD based on different samples such as: complete, first-failure, Type-II PCS and PFFCS. Several works such as Soliman et al. (2012), Abu-Zinadah (2014), Ghitany, Alqallaf and Balakrishnan (2014), Soliman and Al Sobhi (2015), Ahmed (2015), Demġr and Saraçoğlu (2015), Mohan and Chacko (2016), also, Abu-Zinadah and Bakoban (2017) can be obtained as a special cases from the new results. The rest of the paper is organised as follows: In Section 2, the model of PFFCS is described. Section 3 deals with maximum likelihood estimation of the unknown parameter, survival and hazard rate functions of the GGD under PFFCS. Approximate confidence intervals (ACIs) of the unknown parameters and the reliability characteristics are constructed based on the asymptotic normality of the MLEs and the delta method, respectively. An illustrative example with real data is provided in Section 4. Finally, in Section 5, some concluding remarks are provided.

2. Model Description

The PFFCS proposed by Wu and Kuş (2009) to overcome the drawback of the first-failure censoring, that is, that the first-failure censoring does not allow for sets to be removed from the test at the points other than the final termination point. They therefore suggested PFFCS to allow for removal of some of the survival sets from the life-test. The PFFCS is defined as a combination between the concepts of first-failure censoring and Type-II PCS, therefore, an extension of Type-II PCS is referred to as PFFCS. This censoring can be describe as follows: Suppose that n independent groups with k items within each group are put on a life-testing experiment at time zero, m is a pre-fixed number of failures, and the progressive censored sample $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ is pre-fixed such that R_1 groups and the group in which the first failure is observed are randomly removed from the experiment as soon as the first failure (say $X^{\mathbf{R}}_{1:m:n:k}$) has occurred. R_2 groups and the group in which the second failure is observed are randomly removed from the remaining live $n - R_1 - 1$ groups at the time of the second failure (say $X^{\mathbf{R}}_{2:m:n:k}$), and so on. This procedure continues until all remaining live R_m groups and the group in which the mth failure (say $X^{\mathbf{R}}_{m:m:n:k}$) has occurred are removed at the time of the mth failure. Let $X^{\mathbf{R}}_{1:m:n:k}, X^{\mathbf{R}}_{2:m:n:k}, \ldots, X^{\mathbf{R}}_{m:m:n:k}$, be the PFFCS order statistics with the progressive censoring $\mathbf{R}$. If the failure times of the $n \times k$ items originally in the test, where n is the number of independent groups with k items within each group, are from a continuous population with cumulative distribution function (CDF), $F(x;\theta)$, and probability density function (PDF), $f(x;\theta)$, the likelihood function for $X_{(i)}$ instead of $X^{\mathbf{R}}_{i:m:n:k}$, $i = 1, 2, \ldots, m$, is given by

$$L(\theta|data) = CK^m \prod_{i=1}^{m} f(x_{(i)};\theta)[1 - F(x_{(i)};\theta)]^{k(R_i+1)-1}, \tag{1}$$

where $C = n(n - R_1 - 1)(n - R_1 - R_2 - 2) \cdots (n - \sum_{i=1}^{m}(R_i + 1))$ and $n = m + \sum_{i=1}^{m} R_i$.

It should be noted that, the PFFCS (1) is a generalisation of some different samples such as: If putting $\mathbf{R} = (0, 0, \ldots, 0)$, the PFFCS reduces to the joint PDF of the first-failure-censored order statistics. If putting $k = 1$, the PFFCS becomes the joint PDF of Type-II PCS order statistics. If putting $\mathbf{R} = (0, 0, \ldots, n - m)$ and $k = 1$, the PFFCS reduces to Type-II censoring. If putting $\mathbf{R} = (0, 0, \ldots, 0)$ and $k = 1$, then $n = m$ which yields the PFFCS corresponds to the complete sample.

3. Maximum likelihood estimation

In this section, we discuss the MLEs of the three-parameter generalised Gompertz parameters as well as SF and HRF based on PFFCS and construct the corresponding ACIs. Suppose that the lifetime random variable X has a $GGD(\alpha, \lambda, \theta)$ distribution, with shape parameters $\alpha, \lambda > 0$ and scale parameter $\theta \geq 0$, then the CDF, PDF, SF and HRF of X are given, respectively, by

$$F(x;\alpha,\lambda,\theta) = \left[1 - \exp\left(-\frac{\lambda}{\theta}(e^{\theta x} - 1)\right)\right]^{\alpha}, \quad x \geq 0, \tag{2}$$

$$f(x;\alpha,\lambda,\theta) = \alpha\lambda e^{\theta x} \exp\left(-\frac{\lambda}{\theta}(e^{\theta x} - 1)\right)\left[1 - \exp\left(-\frac{\lambda}{\theta}(e^{\theta x} - 1)\right)\right]^{\alpha-1}, \quad x \geq 0, \tag{3}$$

$$S(x;\alpha,\lambda,\theta) = 1 - \left[1 - \exp\left(-\frac{\lambda}{\theta}(e^{\theta x} - 1)\right)\right]^{\alpha}, \quad x \geq 0, \tag{4}$$

$$H(x;\alpha,\lambda,\theta) = \frac{\alpha\lambda e^{\theta x} \exp\left(-\frac{\lambda}{\theta}(e^{\theta x} - 1)\right)\left[1 - \exp\left(-\frac{\lambda}{\theta}(e^{\theta x} - 1)\right)\right]^{\alpha-1}}{1 - \left[1 - \exp\left(-\frac{\lambda}{\theta}(e^{\theta x} - 1)\right)\right]^{\alpha}}, \quad x \geq 0. \tag{5}$$

3.1 Maximum likelihood estimators

Suppose that $n \times k$ independent units taken from a population are placed on a progressively first-failure censored life-test with the corresponding lifetimes being identically distributed having CDF and PDF as defined in (2) and (3), respectively. Substituting (2) and (3) into (1), one can show that

$$\begin{aligned} L(\varpi|data) \propto (\alpha\lambda)^m \exp\left(\theta m\bar{x} - \lambda\sum_{i=1}^{m} W(x_{(i)};\theta)\right) \prod_{i=1}^{m}\left[1 - \exp\left(-\lambda W(x_{(i)};\theta)\right)\right]^{\alpha-1} \\ \times \left[1 - \left[1 - \exp\left(-\lambda W(x_{(i)};\theta)\right)\right]^{\alpha}\right]^{k(R_i+1)-1}, \end{aligned} \tag{6}$$

where $\varpi = (\alpha, \lambda, \theta)^{\mathbf{T}}$ is the parameter vector and $W(x_{(i)}; \theta) = (\exp(x_{(i)}\theta) - 1)/\theta$, $i = 1, 2, \ldots, m$.

Using the general binomial expansion series, (6) is equivalent to

$$L(\varpi|data) \propto (\alpha\lambda)^m \exp\left(\theta m\bar{x} - \lambda \sum_{i=1}^{m} W(x_{(i)}; \theta)\right) \prod_{i=1}^{m} \sum_{d=0}^{k(R_i+1)-1} Q_d \left[1 - \exp\left(-\lambda W(x_{(i)}; \theta)\right)\right]^{\alpha V - 1}, \quad (7)$$

where $V = k(R_i + 1) - d$, $i = 1, 2, \ldots, m$, and $Q_d = (-1)^d \binom{k(R_i+1)-1}{d}$.

The corresponding log-likelihood function of (7), $\ell = \log L$, becomes

$$\ell(\varpi|data) \propto m\log(\alpha\lambda) + \theta m\bar{x} - \lambda \sum_{i=1}^{m} W(x_{(i)}; \theta) + \sum_{i=1}^{m} \log\left(\sum_{d=0}^{k(R_i+1)-1} Q_d \left[1 - \exp\left(-\lambda W(x_{(i)}; \theta)\right)\right]^{\alpha V - 1}\right). \quad (8)$$

Differentiating (8) with respect to α, λ and θ, respectively, we get

$$\frac{\partial \ell(\varpi|data)}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m} \left\{ \frac{\sum_{d=0}^{k(R_i+1)-1} Q_d [1 - \exp(-\lambda W(x_{(i)}; \theta))]^{\alpha V - 1} \log\left[1 - \exp(-\lambda W(x_{(i)}; \theta))\right]}{\sum_{d=0}^{k(R_i+1)-1} Q_d [1 - \exp(-\lambda W(x_{(i)}; \theta))]^{\alpha V - 1}} \right\}, \quad (9)$$

$$\frac{\partial \ell(\varpi|data)}{\partial \lambda} = \frac{m}{\lambda} - \sum_{i=1}^{m} W(x_{(i)}; \theta) + \sum_{i=1}^{m} \left\{ \frac{\sum_{d=0}^{k(R_i+1)-1} Q_d (\alpha V - 1) W(x_{(i)}; \theta) \exp(-\lambda W(x_{(i)}; \theta)) [1 - \exp(-\lambda W(x_{(i)}; \theta))]^{\alpha V - 2}}{\sum_{d=0}^{k(R_i+1)-1} Q_d [1 - \exp(-\lambda W(x_{(i)}; \theta))]^{\alpha V - 1}} \right\}, \quad (10)$$

and

$$\frac{\partial \ell(\varpi|data)}{\partial \theta} = m\bar{x} - \lambda \sum_{i=1}^{m} W'(x_{(i)}; \theta) + \sum_{i=1}^{m} \left\{ \frac{\sum_{d=0}^{k(R_i+1)-1} Q_d (\alpha V - 1) W'(x_{(i)}; \theta) \exp(-\lambda W(x_{(i)}; \theta)) [1 - \exp(-\lambda W(x_{(i)}; \theta))]^{\alpha V - 2}}{\sum_{d=0}^{k(R_i+1)-1} Q_d [1 - \exp(-\lambda W(x_{(i)}; \theta))]^{\alpha V - 1}} \right\}, \quad (11)$$

where $W'(x_{(i)}; \theta) = -[W(x_{(i)}; \theta) - x_{(i)} \exp(\theta x_{(i)})]/\theta$, $i = 1, 2, \ldots, m$. Equating each result in (9)–(11) to zero, three equations must be simultaneously satisfied to obtain the MLEs $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\theta}$ of the three unknown parameters of GGD α, λ and θ, respectively.

It is clear that the MLEs $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\theta}$ cannot be solved analytically and can be obtained by solving the set of nonlinear equations, therefore, numerical methods such as Newton–Raphson can be used to solve these equations. Furthermore, once the estimates of $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\theta}$ are obtained, using the invariance property of MLE, the MLEs of $S(t)$ and $H(t)$, as in (4) and (5), respectively, for a given mission time t can be derived by replacing α, λ and θ by their MLEs $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\theta}$, respectively.

3.2 Approximate interval estimation

Two-sided ACIs for the unknown parameters are constructed based on the asymptotic normal approximation of the maximum likelihood estimators as well as the reliability characteristic such as survival and hazard rate functions are constructed based on the delta method. The Fisher information matrix can be obtained by taking the expectation of the second partial derivatives of the log-likelihood function as

$$I_{ij}(\varpi) = E[-(\partial^2 ln(\theta|\underline{x}/\partial\theta_i\partial\theta_j)], \quad i,j = 1,2,3. \tag{12}$$

Unfortunately, the exact mathematical expressions for the expectation (12) are very difficult to obtain. For convenience, the Fisher information matrix is approximated by

$$I_{ij}(\hat{\varpi}) \cong [-(\partial^2 ln(\theta|\underline{x}/\partial\theta_i\partial\theta_j)]_{\varpi=\hat{\varpi}}, \quad i,j = 1,2,3.$$

which is obtained by dropping E and replacing $\varpi = (\alpha,\lambda,\theta)$ with $\hat{\varpi} = (\hat{\alpha},\hat{\lambda},\hat{\theta})$, respectively, for more detail, see Cohen (1965).

The observed Fisher information matrix has second partial derivatives of (8) as the elements, which easily can be obtained. Hence, the asymptotic variance covariance matrix is obtained by inverting the Fisher information matrix, practically, estimating $I_0^{-1}(\varpi)$ by $I_0^{-1}(\hat{\varpi})$, as

$$I_0^{-1}(\hat{\varpi}) \cong \begin{bmatrix} -\frac{\partial^2\ell(\varpi|data)}{\partial\alpha^2} & -\frac{\partial^2\ell(\varpi|data)}{\partial\alpha\partial\lambda} & -\frac{\partial^2\ell(\varpi|data)}{\partial\alpha\partial\theta} \\ -\frac{\partial^2\ell(\varpi|data)}{\partial\lambda\partial\alpha} & -\frac{\partial^2\ell(\varpi|data)}{\partial\lambda^2} & -\frac{\partial^2\ell(\varpi|data)}{\partial\lambda\partial\theta} \\ -\frac{\partial^2\ell(\varpi|data)}{\partial\theta\partial\alpha} & -\frac{\partial^2\ell(\varpi|data)}{\partial\theta\partial\lambda} & -\frac{\partial^2\ell(\varpi|data)}{\partial\theta^2} \end{bmatrix} = \begin{bmatrix} \hat{\sigma}^2_{\hat{\alpha}} & \hat{\sigma}^2_{\hat{\alpha},\hat{\lambda}} & \hat{\sigma}^2_{\hat{\alpha},\hat{\theta}} \\ \hat{\sigma}^2_{\hat{\lambda},\hat{\alpha}} & \hat{\sigma}^2_{\hat{\lambda}} & \hat{\sigma}^2_{\hat{\lambda},\hat{\theta}} \\ \hat{\sigma}^2_{\hat{\theta},\hat{\alpha}} & \hat{\sigma}^2_{\hat{\theta},\hat{\lambda}} & \hat{\sigma}^2_{\hat{\theta}} \end{bmatrix}. \tag{13}$$

From (6), the elements of the observed information matrix (13) are obtained and reported in the Appendix. Under regularity conditions for the asymptotic properties of the MLEs of the three-parameter GGD, the asymptotic normality of the MLEs $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\theta}$ is approximately multivariate normal with mean ϖ and variance-covariance matrix $I_0^{-1}(\hat{\varpi})$, i.e., $\hat{\varpi} \sim N(\varpi,\ I_0^{-1}(\hat{\varpi}))$, see Lawless (1982).

Hence, the $100(1-\gamma)\%$ two-sided ACIs for the three-parameter GGD α, λ and θ, based on PFFCS are given, respectively, by

$$\hat{\alpha} \pm z_{\gamma/2}\sqrt{\hat{\sigma}^2_{\hat{\alpha}}}, \quad \hat{\lambda} \pm z_{\gamma/2}\sqrt{\hat{\sigma}^2_{\hat{\lambda}}} \quad \text{and} \quad \hat{\theta} \pm z_{\gamma/2}\sqrt{\hat{\sigma}^2_{\hat{\theta}}},$$

where $\hat{\sigma}^2_{\hat{\alpha}}$, $\hat{\sigma}^2_{\hat{\lambda}}$ and $\hat{\sigma}^2_{\hat{\theta}}$, are the elements on the main diagonal of the asymptotic variance-covariance matrix (13), and $z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $(\gamma/2)$.

Furthermore, to construct the asymptotic ACIs of the SF and HRF of GGD based on PFFCS, which are functions of the parameters α, λ and θ. The delta method is considered to obtain the approximate estimates of the variance of $S(t)$ and $H(t)$. The delta method is a general approach for computing ACIs for functions of MLEs, for more detail, see Greene (2003). According to this method, the variances $\hat{\sigma}^2_{\hat{S}(t)}$ and $\hat{\sigma}^2_{\hat{H}(t)}$ of $\hat{S}(t)$ and $\hat{H}(t)$ can be approximated, respectively by

$$\hat{\sigma}^2_{\hat{S}(t)} = [\nabla\hat{S}(t)]^{\mathbf{T}} I_0^{-1}(\hat{\varpi})[\nabla\hat{S}(t)] \quad \text{and} \quad \hat{\sigma}^2_{\hat{H}(t)} = [\nabla\hat{H}(t)]^{\mathbf{T}} I_0^{-1}(\hat{\varpi})[\nabla\hat{H}(t)],$$

where

$$[\nabla\hat{S}(t)]^{\mathbf{T}} = \left[\partial\nabla S(t)/\partial\alpha,\ \partial\nabla S(t)/\partial\lambda,\ \partial\nabla S(t)/\partial\theta\right]_{(\alpha=\hat{\alpha},\ \lambda=\hat{\lambda},\ \theta=\hat{\theta})},$$

Table 1. Failure times of Aarset real data set.

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	45	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

and

$$[\nabla \hat{H}(t)]^{\mathbf{T}} = \left[\partial \nabla H(t)/\partial\alpha,\ \partial \nabla H(t)/\partial\lambda,\ \partial \nabla H(t)/\partial\theta\right]_{(\alpha=\hat{\alpha},\ \lambda=\hat{\lambda},\ \theta=\hat{\theta})}$$

are the gradients of $\hat{S}(t)$ and $\hat{H}(t)$, respectively, with respect to α, λ and θ.

Hence, the $100(1-\gamma)\%$ two-sided ACIs of $S(t)$ and $H(t)$, are given, respectively, by

$$\hat{S}(t) \pm z_{\gamma/2}\sqrt{\hat{\sigma}^2_{\hat{S}(t)}} \quad \text{and} \quad \hat{H}(t) \pm z_{\gamma/2}\sqrt{\hat{\sigma}^2_{\hat{H}(t)}},$$

where $z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $(\gamma/2)$.

4. Real data analysis

In this section we consider a real-life data set representing the lifetimes of 50 devices, originally analysed by Aarset (1987), and use it to illustrate the estimation methods in the preceding section. These lifetimes of the 50 devices are given in Table 1.

Recently, this data set has been analysed by El-Gohary et al. (2013), Abu-Zinadah (2014), Ahmed (2015), Abu-Zinadah and Al-Oufi (2016), Abu-Zinadah and Bakoban (2017). Before progressing further, we first fit the GGD to the complete data set and compare its fitting with two well-established lifetime distributions namely, Gompertz(λ,θ) and exponential(λ) distributions with PDFs given, respectively, by

$$f(x;\lambda,\theta) = \lambda \exp\left(\theta x - \frac{\lambda}{\theta}(e^{\theta x}-1)\right), \quad x \geq 0, \quad \lambda,\theta > 0,$$

and

$$f(x;\lambda) = \lambda e^{-\lambda x}, \quad x \geq 0, \quad \lambda > 0.$$

One question arises about whether the data fit the GGD, GD and ED or not. To check for the goodness of fit, we employed different methods to test the goodness-of-fit of the GGD, GD and ED based on the maximum likelihood estimation method. In order to compare the different lifetime models, we consider the Kolmogorov-Smirnov (K-S) statistics with its p-value, as well as model selection criterion such as: the estimated negative log likelihood function $-\hat{\ell}$, the Akaike information criterion (AIC) proposed by Akaike (1973), defined as AIC $= -2\hat{\ell} + 2S$, the Bayesian information criterion (BIC) proposed by Schwarz (1978), defined as BIC $= -2\hat{\ell} + 2S\log n$, the consistent Akaike information criterion (CAIC), defined as CAIC $= -2\hat{\ell} + 2nS/(n-k-1)$, the Hannan-Quinn information criterion (HQIC), defined as HQIC $= -2\hat{\ell} + 2S\log(\log n)$, where $\hat{\ell}$ denotes the log-likelihood function evaluated at the MLEs, S is the number of model parameters and n is the sample size. The better distribution corresponds to the lowest values of $-\hat{\ell}$, AIC, BIC, CAIC, HQIC and the K-S statistic values and highest p-values. The values of MLEs of the parameters of the ED, GD and GGD reliability models, along with the values of $-\hat{\ell}$, AIC, BIC, CAIC, HQIC and the K-S statistic

Table 2. Summary fit for Aarset data set.

Model	MLE(s)	$-\hat{\ell}$	AIC	BIC	CAIC	HQIC	K-S Statistic	K-S p-value
ED	$\hat{\lambda} = 0.0219$	241.07	484.14	489.96	484.22	484.86	0.191	0.052
GD	$\hat{\lambda} = 0.0097$ $\hat{\theta} = 0.0203$	235.33	474.65	478.47	474.91	476.11	0.169	0.114
GGD	$\hat{\alpha} = 0.5210$ $\hat{\lambda} = 0.0021$ $\hat{\theta} = 0.0481$	**225.07**	**456.15**	**457.97**	**456.67**	**458.33**	**0.141**	**0.269**

Note: Values in bold type represent the best model.

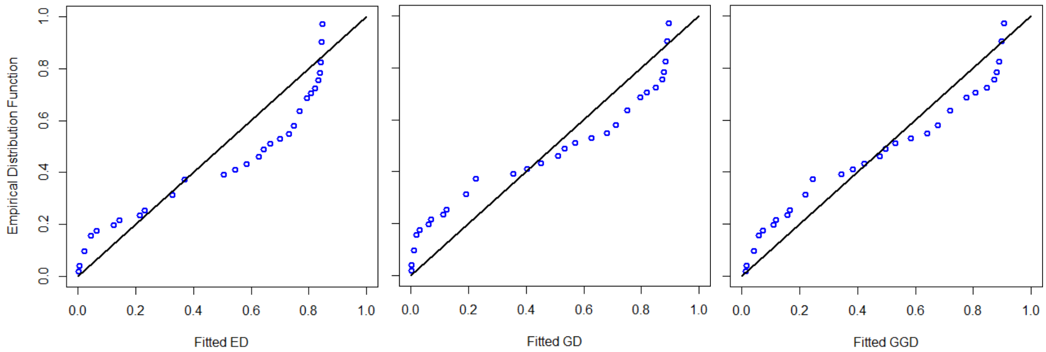

Figure 1. Q-Q plots of ED, GD and GGD reliability models.

with associated p-values are reported in Table 2. The Q-Q plots support the above conclusion. Also, we use a graphical method for goodness-of-fit of distributions by drawing quantile-quantile (Q-Q) plots of the ED, GD and GD reliability models, which are shown in Figure 1. A Q-Q plot depicts the points $\{F^{-1}((i-0.5)/n;\hat{\theta}), x_{(i)}\}$, $i = 1, 2, \ldots, n$, where $\hat{\theta}$ is the MLE of θ.

Since the p-value is much higher than 0.05, we cannot reject the null hypothesis that the data are from the GGD, GD and ED models. Table 2 shows that the GGD is the best choice among the competing reliability models in the literature for fitting Aarset lifetime data, since it has the smallest $-\hat{\ell}$, AIC, BIC, CAIC, HQIC and K-S statistic values and highest p-values. Also, the Q-Q plots support the above findings. For a clearer illustration, Figure 2 shows the fitted CDF and the empirical CDF of ED, GD and GGD, respectively, computed at the estimated parameters. Figure 3 shows the histogram of the real data and the fitted PDF of ED,GD and GGD, respectively, computed at the estimated parameters. To illustrate the inferential method developed in the preceding section, we have assumed that the failure times data of 50 devices are randomly grouped into 25 groups with $k = 2$ devices within each group. The grouped data set is presented in Table 3. Finally, the following first-failure censored sample is obtained in order as: 0.1, 0.2, 1, 1, 1, 1, 2, 3, 6, 12, 18, 18, 18, 32, 36, 45, 63, 67, 67, 67, 72, 82, 83, 84, 84.

Using Tables 1 and 3, the Aarset real data set can be discussed under different generated samples such as:

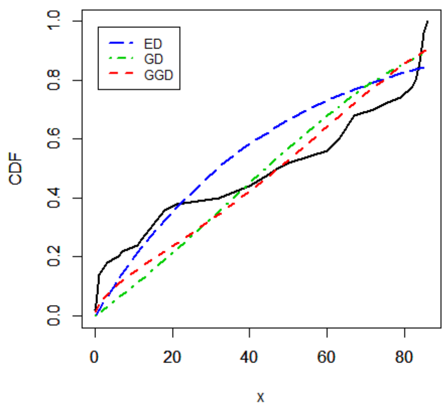

Figure 2. Fitted CDF and the empirical CDFs of reliability models.

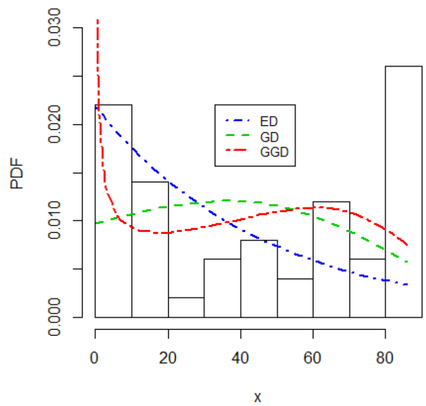

Figure 3. Histogram of real data with fitted PDFs of reliability models.

Table 3. Random grouping of Aarset real data set.

Group item	1	2	3	4	5	6	7	8	9	10	11	12	13
1	21	**1**	**45**	**0.2**	82	85	**18**	**67**	18	45	**0.1**	50	**82**
2	**6**	18	55	40	**36**	**1**	47	84	**12**	**18**	1	**32**	85
Group item	14	15	16	17	18	19	20	21	22	23	24	25	
1	86	**2**	**67**	86	11	67	**67**	**18**	85	**1**	**1**	85	
2	**83**	60	75	**84**	**3**	**63**	79	63	**84**	85	7	**72**	

Note: Bold observations represent the first-failure item in a group.

- Complete sample with $n = m = 50$, $k = 1$ and $\mathbf{R} = (0{*}25)$.
- First-failure censored data with $n = m = 25$, $k = 2$ and $\mathbf{R} = (0{*}25)$.
- Three different samples of Type-II PCS are generated using three different censoring schemes from the failure times data with $n = 50$, $m = 25$ and $k = 1$.
- Three different samples of PFFCSs are generated using three different censoring schemes from the first-failure censored data with $n = 25$, $m = 15$ and $k = 2$.
- The different schemes of Type-II PCS and PFFCS, and corresponding samples are reported in Table 4. For brevity, the censoring scheme $\mathbf{R} = (3, 0, 0, 0, 0, 3)$ is denoted by $\mathbf{R} = (3, 0{*}4, 3)$.

All computations were performed using R statistical programming language software with the reliaR package (developed by Kumar and Ligges, 2011) for plotting and fitting the distribution function, as well as with the maxLik package (developed by Henningsen and Toomet, 2011), which uses the method of Newton–Raphson maximisation in the computations. The maximum likelihood estimates, with corresponding standard errors, the unknown parameters α, λ and θ, as well as the reliability characteristics such as $S(t)$ and $H(t)$, for given $t = 5$, of ED, GD and GGD based on different generated samples such as the complete sample, first-failure censored sample, Type-II PCS

Table 4. Three different generated samples of Type-II PCS and PFFCS.

	Censoring scheme				
Sampling type	n	m	k	**R**	**Sample**
Type-II PCS	50	25	1	$\mathbf{R}_1 = (25, 0^*24)$	0.1, 55, 60, 63, 63, 67, 67, 67,67, 72, 75, 79, 82, 82, 83, 84,84,84, 85, 85, 85, 85, 85, 86, 86.
				$\mathbf{R}_2 = (0^*11, 8, 9, 8, 0^*11)$	0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 36, 67, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.
				$\mathbf{R}_3 = (0^*24, 25)$	0.1, 0.2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 18, 21, 32, 36, 40, 45, 45, 47.
PFFCS	25	15	2	$\mathbf{R}_1 = (10, 0^*14)$	0.1, 18, 18, 32, 36, 45, 63, 67, 67, 67, 72, 82, 83, 84, 84.
				$\mathbf{R}_2 = (0^*6, 3, 4, 3, 0^*6)$	0.1, 0.2, 1, 1, 1, 1, 2, 18, 45, 67, 72, 82, 83, 84, 84.
				$\mathbf{R}_3 = (0^*14, 10)$	0.1, 0.2, 1, 1, 1, 1, 2, 3, 6, 12, 18, 18, 18, 32, 36.

and PFFCS are computed and listed in Table 5. Moreover, the 95% two-sided ACIs with associated lengths of the maximum likelihood estimates are obtained and listed in Table 6.

5. Concluding remarks

In this paper we consider the problem of estimating the unknown parameters and the reliability characteristics of GGD based on PFFCS. Maximum likelihood estimates of GGD were compared with GD and ED under different samples such as: complete sample, first-failure censored sample, Type-II PCS and PFFCS. In addition, the 95% two-sided ACIs of the unknown parameters as well as the reliability characteristics are constructed. It is observed that the MLEs cannot be obtained in closed form, but can be computed and evaluated numerically. Therefore a numerical example with the Aarset real data set has been presented to illustrate the inferential results established here. Moreover, we generalized some results in several works which may be obtained as special cases from the new results such as: Soliman et al. (2012) and Soliman and Al Sobhi (2015) if putting $\alpha = 1$; Abu-Zinadah (2014) if putting $\mathbf{R} = (0, 0, \ldots, n - m)$ and $k = 1$; Ghitany et al. (2014), Mohan and Chacko (2016) if putting $\alpha = k = 1$; also, Ahmed (2015), Demġr and Saraçoğlu (2015), Abu-Zinadah and Bakoban (2017) if putting $k = 1$. From Table 5 it can be seen that the maximum likelihood estimates of the unknown parameters α, λ and θ, as well as reliability characteristic $S(t)$ and $H(t)$ are very good in terms of the standard errors. Also, Table 6 shows that the corresponding lengths of the two-sided 95% ACIs are shortest.

Appendix: Fisher's elements

From (6), the elements of the observed Fisher information matrix (13) become:

$$\frac{\partial^2 \ell(\varpi|data)}{\partial\alpha^2} = -\frac{m}{\alpha^2} - \sum_{i=1}^{m}\{(k(R_i + 1) - 1)(Z(x_{(i)}; \lambda, \theta))^{\alpha}(\log Z(x_{(i)}; \lambda, \theta))^2(1 - (Z(x_{(i)}; \lambda, \theta))^{\alpha})^{-2}\},$$

Table 5. The maximum likelihood estimates with corresponding standard errors of the unknown parameters and the reliability characteristics of ED, GD and GGD in the data obtained from complete sampling, first-failure censored sampling, Type-II PCS and PFFCS.

		Censored sample							
				Type-II PCS			PFFCS		
Model		**Complete**	**First failure**	$\mathbf{R}_1$	$\mathbf{R}_2$	$\mathbf{R}_3$	$\mathbf{R}_1$	$\mathbf{R}_2$	$\mathbf{R}_3$
ED	$\hat{\lambda}$	0.0219	0.0012	0.0136	0.0124	0.0159	0.0183	0.0099	0.0405
		0.0031	0.0002	0.0027	0.0025	0.0032	0.0047	0.0026	0.0105
	$\hat{S}(t)$	0.8963	0.9942	0.9341	0.9399	0.9238	0.9125	0.9515	0.8168
		0.0139	0.0012	0.0127	0.0012	0.0146	0.0216	0.0122	0.0423
	$\hat{H}(t)$	0.0219	0.0012	0.0136	0.0124	0.0159	0.0183	0.0099	0.0405
		0.0031	0.0002	0.0027	0.0025	0.0032	0.0047	0.0026	0.0105
GD	$\hat{\lambda}$	0.0097	0.0010	0.0022	0.0065	0.0033	0.0014	0.0074	0.0026
		0.0030	0.0004	0.0005	0.0027	0.0014	0.0010	0.0035	0.0008
	$\hat{\theta}$	0.0203	0.0042	0.0002	0.0171	0.0534	0.0422	0.0089	0.0001
		0.0060	0.0088	0.0119	0.0083	0.0137	0.0127	0.0109	0.0141
	$\hat{S}(t)$	0.9501	0.9949	0.9891	0.9667	0.9814	0.9922	0.9631	0.9873
		0.0143	0.0878	0.0023	0.0136	0.0075	0.0299	0.0067	0.0038
	$\hat{H}(t)$	0.0108	0.0010	0.0022	0.0071	0.0043	0.0017	0.0077	0.0025
		0.0030	0.0089	0.0005	0.0028	0.0016	0.0175	0.0013	0.0007
GGD	$\hat{\alpha}$	0.8635	0.7995	1.0129	0.5496	0.2547	0.4790	0.3413	0.3449
		0.0269	0.0292	0.2200	0.1001	0.0463	0.1280	0.0834	0.0835
	$\hat{\lambda}$	0.0001	0.0004	0.0003	0.0056	0.0001	0.0001	0.0002	0.0005
		0.0001	0.0002	0.0004	0.0026	0.0001	0.0001	0.0001	0.0003
	$\hat{\theta}$	0.0957	0.0133	0.0699	0.0014	0.0771	0.0755	0.0528	0.0406
		0.0367	0.0097	0.0177	0.0102	0.0239	0.0110	0.0154	0.0374
	$\hat{S}(t)$	0.9983	0.9935	0.9984	0.8603	0.8431	0.9712	0.9009	0.8614
		0.0007	0.0033	0.0025	0.0496	0.0518	0.0268	0.0546	0.0637
	$\hat{H}(t)$	0.0004	0.0011	0.0004	0.0177	0.0114	0.0034	0.0085	0.0122
		0.0011	0.0005	0.0006	0.0179	0.0019	0.2006	0.4520	0.5380

Note: The second row of each estimate represent the corresponding standard error.

Table 6. The 95% two-sided ACIs for the unknown parameters and reliability characteristics of ED, GD and GGD in the data obtained from complete sampling, first-failure censored sampling, Type-II PCS and PFFCS.

					Censored sample					
					Type-II PCS			PFFCS		
Model			**Complete**	**First failure**	$\mathbf{R}_1$	$\mathbf{R}_2$	$\mathbf{R}_3$	$\mathbf{R}_1$	$\mathbf{R}_2$	$\mathbf{R}_3$
ED	λ	Lower	0.0158	0.0007	0.0083	0.0075	0.0096	0.0090	0.0049	0.0199
		Upper	0.0279	0.0016	0.0189	0.0173	0.0221	0.0276	0.0150	0.0609
		Length	0.0121	0.0009	0.0106	0.0098	0.0125	0.0186	0.0101	0.0410
	$S(t)$	Lower	0.8691	0.9920	0.9091	0.9197	0.8951	0.8702	0.9276	0.7331
		Upper	0.9235	0.9965	0.9591	0.9627	0.9525	0.9548	0.9754	0.9004
		Length	0.0544	0.0045	0.0500	0.0456	0.0574	0.0846	0.0478	0.1673
	$H(t)$	Lower	0.0158	0.0007	0.0083	0.0075	0.0096	0.0090	0.0049	0.0199
		Upper	0.0279	0.0016	0.0189	0.0173	0.0221	0.0276	0.0150	0.0609
		Length	0.0121	0.0009	0.0106	0.0098	0.0125	0.0186	0.0101	0.0410
GD	λ	Lower	0.0038	0.0003	0.0012	0.0013	0.0005	0.0001	0.0006	0.0009
		Upper	0.0156	0.0017	0.0031	0.0117	0.0061	0.0034	0.0141	0.0041
		Length	0.0118	0.0014	0.0019	0.0104	0.0056	0.0033	0.0135	0.0032
	θ	Lower	0.0085	0.0000	0.0000	0.0008	0.0266	0.0173	0.0000	0.0000
		Upper	0.0320	0.0214	0.0236	0.0334	0.0803	0.0670	0.0304	0.0278
		Length	0.0235	0.0214	0.0236	0.0326	0.0537	0.0497	0.0304	0.0278
	$S(t)$	Lower	0.9219	0.8229	0.9846	0.9399	0.9668	0.9336	0.9501	0.9800
		Upper	0.9782	0.9999	0.9937	0.9934	0.9959	0.9999	0.9761	0.9947
		Length	0.0563	0.1770	0.0091	0.0535	0.0291	0.0663	0.0260	0.0147
	$H(t)$	Lower	0.0048	0.0000	0.0013	0.0016	0.0011	0.0000	0.0000	0.0011
		Upper	0.0167	0.0186	0.0031	0.0120	0.0074	0.0324	0.0299	0.0039
		Length	0.0119	0.0186	0.0018	0.0110	0.0063	0.0324	0.0299	0.0028
GGD	α	Length	0.8107	0.6174	0.5815	0.3526	0.1638	0.2286	0.1779	0.1814
		Length	0.9163	0.9816	1.4445	0.7465	0.3456	0.7294	0.5048	0.5085
		Length	0.1056	0.3642	0.8630	0.3939	0.1818	0.5008	0.3269	0.3271
	λ	Lower	0.0000	0.0000	0.0000	0.0006	0.0000	0.0000	0.0000	0.0000
		Upper	0.0001	0.0007	0.0010	0.0107	0.0003	0.0002	0.0004	0.0012
		Length	0.0001	0.0007	0.0010	0.0101	0.0003	0.0002	0.0004	0.0012
	θ	Lower	0.0238	0.0000	0.0352	0.0000	0.0303	0.0539	0.0227	0.0000
		Upper	0.1676	0.0323	0.1045	0.0213	0.1239	0.0970	0.0829	0.1139
		Length	0.1438	0.0323	0.0693	0.0213	0.0936	0.0431	0.0602	0.1139
	$S(t)$	Lower	0.9969	0.9870	0.9935	0.7632	0.7415	0.9186	0.7939	0.7364
		Upper	0.9996	0.9999	1.0000	0.9575	0.9447	1.0000	1.0000	0.9864
		Length	0.0027	0.0129	0.0065	0.1943	0.2032	0.0814	0.2061	0.2500
	$H(t)$	Lower	0.0002	0.0000	0.0000	0.0000	0.0076	0.0000	0.0000	0.0000
		Upper	0.0006	0.0022	0.0015	0.3689	0.0152	0.3957	0.8954	1.0664
		Length	0.0004	0.0022	0.0015	0.3689	0.0076	0.3957	0.8954	1.0664

$$\begin{aligned}\frac{\partial^2 \ell(\varpi|data)}{\partial \lambda^2} = & -\frac{m}{\lambda^2}-(\alpha-1)\sum_{i=1}^{m}\Big\{(W(x_{(i)};\theta))^2(Z(x_{(i)};\lambda,\theta))^{-2}\exp(-\lambda W(x_{(i)};\theta))\Big\}\\ & -\alpha\sum_{i=1}^{m}\Big\{(k(R_i+1)-1)(W(x_{(i)};\theta))^2(Z(x_{(i)};\lambda,\theta))^{\alpha-1}\big(1-(Z(x_{(i)};\lambda,\theta))^{\alpha}\big)^{-2}\\ & \times\exp(-\lambda W(x_{(i)};\theta))\big[\exp(-\lambda W(x_{(i)};\theta))(Z(x_{(i)};\lambda,\theta))^{-1}((Z(x_{(i)};\lambda,\theta))^{\alpha}+\alpha-1)\\ & + Z(x_{(i)};\lambda,\theta))^{\alpha}-1\big]\Big\},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ell(\varpi|data)}{\partial \theta^2} = & -\lambda\sum_{i=1}^{m}(W''(x_{(i)};\theta))^2+(\alpha-1)\lambda\sum_{i=1}^{m}(Z(x_{(i)};\lambda,\theta))^{-2}\{Z(x_{(i)};\lambda,\theta)\exp(-\lambda W(x_{(i)};\theta))\\ & \times[W''(x_{(i)};\theta)-\lambda(W'(x_{(i)};\theta))^2]-\lambda(W'(x_{(i)};\theta))^2\exp(-2\lambda W(x_{(i)};\theta))\}\\ & -\alpha\lambda\sum_{i=1}^{m}(k(R_i+1)-1)(1-Z(x_{(i)};\lambda,\theta))^{\alpha})^{-2}\{(1-(Z(x_{(i)};\lambda,\theta)^{\alpha})\{(\alpha-1)\\ & \times(Z(x_{(i)};\lambda,\theta)^{(\alpha-2)}(\lambda(W'(x_{(i)};\theta)))^2\exp(-2\lambda W(x_{(i)};\theta))+\lambda(Z(x_{(i)};\lambda,\theta)^{(\alpha-1)}\\ & \times\exp(-\lambda W(x_{(i)};\theta))[W''(x_{(i)};\theta)-\lambda(W'(x_{(i)};\theta))^2]\}+\alpha(Z(x_{(i)};\lambda,\theta)^{2(\alpha-1)}\\ & \times(\lambda(W'(x_{(i)};\theta)))^2\exp(-2\lambda W(x_{(i)};\theta))\},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ell(\varpi|data)}{\partial \alpha\partial\lambda} = & \sum_{i=1}^{m}\{(Z(x_{(i)};\lambda,\theta))^{-1}W(x_{(i)};\theta)\exp(-\lambda W(x_{(i)};\theta))\}-\sum_{i=1}^{m}\{(k(R_i+1)-1)W(x_{(i)};\theta)\\ & \times(Z(x_{(i)};\lambda,\theta))^{\alpha-1}(1-(Z(x_{(i)};\lambda,\theta))^{\alpha})^{-2}\exp(-\lambda W(x_{(i)};\theta))[1-(Z(x_{(i)};\lambda,\theta))^{\alpha}\\ & +\alpha\log Z(x_{(i)};\lambda,\theta)]\},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ell(\varpi|data)}{\partial \alpha\partial\theta} = & \sum_{i=1}^{m}\{(Z(x_{(i)};\lambda,\theta))^{-1}W'(x_{(i)};\theta)\exp(-\lambda W(x_{(i)};\theta))\}-\lambda\sum_{i=1}^{m}\{(k(R_i+1)-1)W'(x_{(i)};\theta)\\ & \times(Z(x_{(i)};\lambda,\theta))^{\alpha-1}(1-(Z(x_{(i)};\lambda,\theta))^{\alpha})^{-2}\exp(-\lambda W(x_{(i)};\theta))[1-(Z(x_{(i)};\lambda,\theta))^{\alpha}\\ & +\alpha\log Z(x_{(i)};\lambda,\theta)]\},\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial^2 \ell(\varpi|data)}{\partial \lambda\partial\theta}\\ = & -\sum_{i=1}^{m}W(x_{(i)};\theta)+(\alpha-1)\sum_{i=1}^{m}(Z(x_{(i)};\lambda,\theta))^{-2}\{\lambda W(x_{(i)};\theta)W'(x_{(i)};\theta)\exp(-\lambda W(x_{(i)};\theta))\\ & \times[Z(x_{(i)};\lambda,\theta))-\exp(-\lambda W(x_{(i)};\theta))-1]+(1-Z(x_{(i)};\lambda,\theta))W'(x_{(i)};\theta)\\ & \times\exp(-\lambda W(x_{(i)};\theta))\}-\alpha\sum_{i=1}^{m}(k(R_i+1)-1)(1-(Z(x_{(i)};\lambda,\theta))^{\alpha})^{-2}\{(1-(Z(x_{(i)};\lambda,\theta))^{\alpha})\\ & \times(Z(x_{(i)};\lambda,\theta))^{\alpha-1}W'(x_{(i)};\theta)\exp(-\lambda W(x_{(i)};\theta))[1+\lambda W(x_{(i)};\theta)\exp(-\lambda W(x_{(i)};\theta))\\ & (1+(\alpha-1)(Z(x_{(i)};\lambda,\theta))^{-1})]+\alpha(Z(x_{(i)};\lambda,\theta))^{2(\alpha-1)}\lambda W(x_{(i)};\theta)W'(x_{(i)};\theta)\\ & \times\exp(-2\lambda W(x_{(i)};\theta))\},\end{aligned}$$

where $W''(x_{(i)};\theta) = -2[W'(x_{(i)};\theta) - \frac{1}{2}x_{(i)}^2 \exp(\theta x_{(i)})]/\theta$, and $Z(x_{(i)};\lambda,\theta) = 1 - \exp(-\lambda W(x_{(i)};\theta))$, $i = 1,2,\ldots,m$.

Acknowledgements. The authors would like to express thanks to the Editors and the anonymous referees for their valuable comments and suggestions, which significantly improved the paper.

References

Aarset, M. V. (1987). How to identify a bathtub hazard rate. *IEEE Transactions on Reliability*, **36**, 106–108.

Abu-Zinadah, H. H. (2014). Bayesian estimation on the exponentiated Gompertz distribution under type II censoring. *International Journal of Contemporary Mathematical Sciences*, **9**, 497–505.

Abu-Zinadah, H. H. and Al-Oufi, A. S. (2016). Different method estimations of the three parameters of exponentiated Gompertz distribution. *Applied Mathematics & Information Sciences*, **10**, 705–710.

Abu-Zinadah, H. H. and Bakoban, R. (2017). Bayesian estimation of exponentiated Gompertz distribution under progressive censoring type-II. *Journal of Computational and Theoretical Nanoscience*, **14**, 5239–5247.

Ahmed, E. A. (2015). Estimation of some lifetime parameters of generalized Gompertz distribution under progressively type-II censored data. *Applied Mathematical Modelling*, **39**, 5567–5578.

Akaike, H. (1973). Information theory and an extension of maximum likelihood principle. *In* Petrov, B. N. and Caski, F. (Editors) *Proceedings of the Second International Symposium on Information Theory*. 267–281.

Balakrishnan, N. and Cramer, E. (2014). *The Art of Progressive Censoring*. Springer, New York.

Cohen, A. C. (1963). Progressively censored samples in life testing. *Technometrics*, **5**, 327–339.

Cohen, A. C. (1965). Maximum likelihood estimation in the Weibull distribution based on complete and on censored samples. *Technometrics*, **7**, 579–588.

Demġr, E. and Saraçoğlu, B. (2015). Maximum likelihood estimation for the parameters of the generalized Gompertz distribution under progressive type-II right censored samples. *Journal of Selcuk University Natural and Applied Science*, **4**, 41–48.

El-Gohary, A., Alshamrani, A., and Al-Otaibi, A. N. (2013). The generalized Gompertz distribution. *Applied Mathematical Modelling*, **37**, 13–24.

Ghitany, M., Alqallaf, F., and Balakrishnan, N. (2014). On the likelihood estimation of the parameters of Gompertz distribution based on complete and progressively type-II censored samples. *Journal of Statistical Computation and Simulation*, **84**, 1803–1812.

Gompertz, B. (1825). XXIV. On the nature of the function expressive of the law of human mortality, and on a new mode of determining the value of life contingencies. In a letter to Francis Baily, Esq. FRS and others. *Philosophical Transactions of the Royal Society of London*, **115**, 513–583.

Greene, W. H. (2003). *Econometric Analysis*. Fifth edition. Prentice Hall, Upper Saddle River, New Jersey.

Henningsen, A. and Toomet, O. (2011). maxLik: A package for maximum likelihood estimation in R. *Computational Statistics*, **26**, 443–458.

KUMAR, V. AND LIGGES, U. (2011). *reliaR: Package for some probability distributions.* URL: *https://CRAN.R-project.org/package=reliaR*

LAWLESS, J. F. (1982). *Statistical Models and Methods for Lifetime Data*. Wiley & Sons, New York.

MOHAN, R. AND CHACKO, M. (2016). Estimation of parameters of Gompertz distribution under progressive type-II censoring. *The Kerala Statistical Association*, **27**, 84–103.

MUDHOLKAR, G. S., SRIVASTAVA, D. K., AND FREIMER, M. (1995). The exponentiated Weibull family: A reanalysis of the bus-motor-failure data. *Technometrics*, **37**, 436–445.

SCHWARZ, G. (1978). Estimating the dimension of a model. *The Annals of Statistics*, **6**, 461–464.

SOLIMAN, A. A., ABD-ELLAH, A. H., ABOU-ELHEGGAG, N. A., AND ABD-ELMOUGOD, G. A. (2012). Estimation of the parameters of life for Gompertz distribution using progressive first-failure censored data. *Computational Statistics & Data Analysis*, **56**, 2471–2485.

SOLIMAN, A. A. AND AL SOBHI, M. M. (2015). Bayesian MCMC inference for the Gompertz distribution based on progressive first-failure censoring data. *In AIP Conference Proceedings*, volume 1643. AIP, 125–134.

WU, S.-J. AND KUŞ, C. (2009). On estimation based on progressive first-failure-censored sampling. *Computational Statistics & Data Analysis*, **53**, 3659–3670.

Manuscript received 2018-01-14, revised 2018-06-06, accepted 2018-08-04.