South African Statistical Journal
2019, Vol. 53, No. 1, 31–34

ON A CHARACTERISTIC PROPERTY OF DISTRIBUTIONS RELATED TO THE LAPLACE

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It is shown that the asymmetric Laplace distribution uniquely arises as the distribution of both a difference between independent positive random variables and of a random choice between those random variables, one of them having been given a negative sign. Related results on the positive half-line are also given.

Key words: Asymmetric Laplace distribution, Double Pareto distribution, Ratios of random variables.

1. Introduction

Ignoring its location parameter for the theoretical purposes of this note, the asymmetric Laplace distribution on $\mathbb{R}$ has density

$$f(x) = \frac{\alpha\beta}{\alpha+\beta}\begin{cases} e^{\beta x} & \text{for } x < 0, \\ e^{-\alpha x} & \text{for } x \geq 0, \end{cases}$$

with $\alpha, \beta > 0$. See Chapter 3 of Kotz, Kozubowski and Podgórski (2001) for a detailed treatment in the alternative parameterisation $\kappa = \sqrt{\alpha/\beta}$, $\sigma = \sqrt{2/\alpha\beta}$. Of course, the asymmetric Laplace distribution becomes the usual Laplace distribution when $\alpha = \beta$.

Among its many representations (Kotz et al., 2001, Table 3.4), an asymmetric Laplace random variable X can be written as either

1. $X = \begin{cases} -E/\beta & \text{with probability } \alpha/(\alpha+\beta), \\ E/\alpha & \text{with probability } \beta/(\alpha+\beta), \end{cases}$

   where E is a unit exponential random variable; or

2. $X = \dfrac{E_1}{\alpha} - \dfrac{E_2}{\beta}$,

   where E_1 and E_2 are independent unit exponential random variables.

That is, an asymmetric Laplace random variable arises both as a difference between independent exponential random variables with suitable rates and from a random choice between such exponential random variables, one of them having been given a negative sign.

MSC2010 subject classifications. 62E10.

In this short article, I address the question of how common is this kind of property, equivalence between random signing and taking a difference. The answer is that it is uncommon: the asymmetric Laplace distribution is characterised by this equivalence. To see this, I find it more transparent to work in terms of $Y = e^{-X}$, a random variable with the log-asymmetric-Laplace distribution on $\mathbb{R}^+$ (Kozubowski and Podgórski, 2003), to provide the corresponding results on that support, and then to translate back to $\mathbb{R}$ via $X = -\log Y$. The arguments leading up to the log-asymmetric-Laplace distribution are given in Section 2 and their translation back to the asymmetric Laplace distribution in Section 3. Potential application to goodness-of-fit testing is also outlined in Section 3.

2. On ratios of independent $(0,1)$-support random variables

Consider first the distribution of $W = U/V$ where $U \sim U(0,1)$ independently of $V \sim U(0,1)$, where '$\sim$' denotes 'is distributed as' and $U(0,1)$ is the uniform distribution on $(0,1)$. Note that W has the same distribution as $1/W = V/U$. It is easy to see that the density of W is

$$f_W(w) = \frac{1}{2}\begin{cases} 1 & \text{for } 0 < w < 1, \\ 1/w^2 & \text{for } w \geq 1. \end{cases}$$

This is equivalent to saying that

$$W = \begin{cases} U & \text{with probability } 1/2, \\ 1/U & \text{with probability } 1/2. \end{cases}$$

Equivalently again, conditional on $U < V$, $W = U/V$ has the same distribution as U and, conditional on $U \geq V$, $W = U/V$ has the same distribution as $1/V$ (or $1/U$).

Elsewhere, I have called this result remarkable, but how remarkable is it? Consider the distribution of $Z = R/S$, where $R \sim g$ independently of $S \sim h$, with g and h being the densities of arbitrary distributions on $(0,1)$. It is a little easier to work with c.d.f.s (so let G and H be their respective c.d.f.s), in which case the c.d.f. of Z is

$$F_Z(z) = \begin{cases} \int_0^1 G(zs)h(s)ds & \text{for } 0 < z < 1, \\ \int_0^{1/z} G(zs)h(s)ds + 1 - H(1/z) & \text{for } z \geq 1. \end{cases}$$

This says that, conditional on $R < S$, $Z = R/S$ has the distribution with c.d.f. $\int_0^1 G(zs)h(s)ds / \int_0^1 G(s)h(s)ds$ on $(0,1)$ while, conditional on $R \geq S$, Z has the distribution with c.d.f.

$$\frac{\int_0^{1/z} G(zs)h(s)ds + 1 - H(1/z) - \int_0^1 G(s)h(s)ds}{1 - \int_0^1 G(s)h(s)ds}$$

on $[1,\infty)$. Here, note that $\int_0^1 G(s)h(s)ds = P(R < S)$.

Now concentrate on the conditional distribution of Z given that $R < S$. When is it equal to the distribution of R?

Theorem 1 *For $0 < z < 1$ and h any density on $(0,1)$, $G_\alpha(z) = z^\alpha$, for any $\alpha > 0$, are the only c.d.f. solutions to $G(z) = \int_0^1 G(zs)h(s)ds / \int_0^1 G(s)h(s)ds$.*

Proof. We need $\int_0^1 G(zs)h(s)ds = \int_0^1 G(z)G(s)h(s)ds$ for all densities h on $(0,1)$. This can only happen if $G(zs) = G(z)G(s)$ for all $0 < z, s < 1$. This functional equation is satisfied only by powers, of which only the positive powers yield valid c.d.f.'s. ■

G_α is the c.d.f. of a power law distribution (the Beta$(\alpha, 1)$ distribution) with density $\alpha z^{\alpha-1}$ on $0 < z < 1$, and is the distribution of $U^{1/\alpha}$ where $U \sim U(0,1)$.

So when, additionally, is the conditional distribution of Z given that $R > S$ equal to the distribution of $1/S$? This can be answered immediately by consideration of the distribution of $1/Z = S/R$, leading to the following characterisation result.

Corollary 1 *Let R and S be independent random variables on $(0,1)$. The only distributions such that, conditional on $R < S$, $Z = R/S$ has the same distribution as R and, conditional on $R \geq S$, $Z = R/S$ has the same distribution as $1/S$ are the distributions of $Z_{\alpha\beta} = U^{1/\alpha}/V^{1/\beta}$ where $U \sim U(0,1)$ independently of $V \sim U(0,1)$ and $\alpha, \beta > 0$. These have density*

$$f_{\alpha\beta}(z) = \frac{\alpha\beta}{\alpha+\beta}\begin{cases} z^{\alpha-1} & \text{for } 0 < z < 1, \\ 1/z^{\beta+1} & \text{for } z \geq 1, \end{cases}$$

which is equivalent to saying that

$$Z_{\alpha\beta} = \begin{cases} U^{1/\alpha} & \text{with probability } \beta/(\alpha+\beta), \\ 1/V^{1/\beta} & \text{with probability } \alpha/(\alpha+\beta). \end{cases}$$

The density $f_{\alpha\beta}$ is that of the log-asymmetric-Laplace distribution (Kozubowski and Podgórski, 2003) with unit scale. Since reciprocals of powers of uniform random variables have Pareto distributions and ratios of power law random variables are ratios of Pareto random variables with power parameters swapped, this distribution is also called a double Pareto distribution (Reed, 2001).

3. On differences of independent positive-support random variables

The random variables $P = -\log R$ and $Q = -\log S$ have arbitrary distributions on $\mathbb{R}^+$. Corollary 1 can therefore be converted directly into the required result about asymmetric Laplace distributions simply by taking logs. Note that, of course, the distribution of $-\log U$ is the unit exponential distribution.

Corollary 2 *Let P and Q be independent random variables on $\mathbb{R}^+$. The only distributions such that, conditional on $P > Q$, $Y = P - Q$ has the same distribution as P and, conditional on $P \leq Q$, $Y = P - Q$ has the same distribution as $-Q$ are the distributions of $Y_{\alpha\beta} = -\frac{1}{\alpha}\log U + \frac{1}{\beta}\log V = \frac{1}{\alpha}E_1 - \frac{1}{\beta}E_2$ where E_1 and E_2 are unit exponential random variables (as in Item 2 of the Introduction). These have the asymmetric Laplace density given at the start of the article and are equivalent to it being the case that*

$$Y_{\alpha\beta} = \begin{cases} E/\alpha & \text{with probability } \beta/(\alpha+\beta), \\ -E/\beta & \text{with probability } \alpha/(\alpha+\beta), \end{cases}$$

where E is a unit exponential random variable (as in Item 1 of the Introduction).

Corollary 2 can potentially be used to test the goodness-of-fit of the asymmetric Laplace distribution to a set of data. First, fit the asymmetric Laplace distribution, with location parameter θ say, included, to the data; write the estimated parameter values as $\hat{\theta}$, $\hat{\alpha}$ and $\hat{\beta}$. Second, subtract $\hat{\theta}$ from each data value. Third, partition the dataset, thus relocated, into two subsets, X_- say, containing those values less then 0 and X_+ say, containing those values greater than 0; and take absolute values of the members of X_-. Fourth, generate some number N of pseudo-samples, each pseudo-sample being obtained by randomly selecting one value from within X_- and multiplying it by $\hat{\beta}$, randomly selecting one value from within X_+ and multiplying it by $\hat{\alpha}$, and calculating the distance between the two. Fifth, generate N further, different, pseudo-samples, each pseudo-sample being obtained this time by first selecting a value at random from the recombined sample (X_-, X_+), and scaling it as above according to the subset of the data in which it lies. Sixth, use as test statistic some measure of the distance between the distributions of the two sets of pseudo-samples generated as above. Finally, obtain the sampling distribution of the test statistic by bootstrapping the whole procedure.

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Manuscript received 2018-10-29, revised 2019-01-09, accepted 2019-01-17.