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ASYMPTOTIC NORMALITY OF THE LOCAL LINEAR ESTIMATION OF THE CONDITIONAL DENSITY FOR FUNCTIONAL DEPENDENT AND CENSORED DATA

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In this paper, we study the asymptotic behaviour of the nonparametric local linear estimation of the conditional density when the interest variable is subject to random right censoring of the scalar response variable given a functional random covariate taking values in a semi-metric space. Under some regularity conditions, the joint asymptotic normality of the estimators of the conditional density is established.

Key words: Asymptotic normality, Conditional density, Functional censored data, Local linear estimator, Mixing data, Small ball probability.

1. Introduction

The conditional density function plays an important role in nonparametric prediction. In addition, it provides the most informative summary of the relationships between a variable of interest Y and a covariate X. There is extensive literature on conditional density function estimation when the data are either independent or dependent and in finite or infinite dimensional spaces. For example, Hyndman et al. (1996) studied the kernel estimator of the conditional density and its bias-corrected version. Fan et al. (1996) developed a direct estimation method via an innovative ‘double-kernel’ local linear approach. Bashtannyk and Hyndman (2001) and Hyndman and Yao (2002) proposed several simple and useful rules for selecting bandwidths for conditional density estimation. Hall et al. (2004) applied the cross-validation technique to estimate the conditional density. Fan and Yim (2004) proposed a consistent data-driven bandwidth selection procedure in estimating the conditional density functions. In these papers, it is assumed that the data are fully observed.

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In the case of finite-dimensional data, it is well known that the kernel method is inferior to the local linear fitting because of limitations such as large bias, non-adaptation as well as boundary effects. Recently, some results on the local linear modelling in the functional data setting have been obtained. Baíllo and Grané (2009) first proposed a local linear estimator of the regression operator when the explanatory variable takes values in a Hilbert space. When the explanatory variable takes values in a semi-metric space, Barrientos-Marin et al. (2010) proposed another alternative version of the local linear estimator of the regression operator in the i.i.d. setup, which was called a locally modelled regression estimator. They found that the estimator made its computation easy and fast while keeping good performance. Then, this method has been employed to estimate the conditional density (Demongeot et al., 2013; Rachdi et al., 2014), the conditional distribution (see Demongeot et al., 2014) and the conditional quantile (see Messaci et al., 2015) of a scalar response given a functional explanatory variable in the i.i.d. setting.

In some fields such as reliability or survival analysis, the random variable (rv) Y (which has unknown continuous distribution function (df) F) can be regarded as the lifetime of patients under study. In reality it is not possible to observe the survival time of all patients, and often some of them are still alive at the end of the study, withdraw, or die from other causes than those addressed by the study. In those cases, we observe another random variable C indicating censoring. Then, assuming that $\{Y_i, i \geq 1\}$ is a stationary sequence satisfying some dependency conditions, and $\{C_i, i \geq 1\}$ is a sequence of i.i.d. censoring random variables with common unknown continuous distribution function (df) G and we observe only the n pairs (T_i, δ_i), for $i = 1, \ldots, n$, where $T_i = \min(Y_i, C_i)$ and $\delta_i = I(Y_i \leq C_i)$. We suppose that (Y_i) and (C_i) for $i = 1, \ldots, n$ are independent, which ensures the identifiability of the model. Moreover, in this case the distribution J of T_1 satisfies $1 - J = (1 - F)(1 - G)$.

Now let X be a rv taking values in $\mathcal{F}$, where $\mathcal{F}$ is a semi-metric space equipped with a semi-metric d. A semi-metric space $(\mathcal{F}, d)$ satisfies all the conditions of a metric space except it need not satisfy $d(x_1, x_2) = 0, x_1, x_2 \in \mathcal{F} \Rightarrow x_1 = x_2$. For example, set $\mathcal{F}_m = C^m[0,1]$, i.e., the set of functions with continuous mth derivatives on $[0,1], m \geq 0$; the semi-metric $d_m(\cdot,\cdot)$ is defined as

$$d_m(x_1, x_2) = \left\{ \int_0^1 [x_1^{(m)}(t) - x_2^{(m)}(t)]^2 dt \right\}^{1/2}, \quad x_1, x_2 \in \mathcal{F}_m,$$

where $x_1^{(m)}(\cdot)$ and $x_2^{(m)}(\cdot)$ denote the mth derivatives of $x_1(\cdot)$ and $x_2(\cdot)$, respectively. In the case that $m = 3$, it is easily seen that if $x_1(t) = t, x_2(t) = t^2, t \in [0,1]$, then $d_3(x_1, x_2) = 0$. Then, $(\mathcal{F}_3, d_3)$ is a semi-metric space .

The *α-mixing (strong mixing)* condition is the weakest among mixing conditions known in the literature and it has an important role in a number of applications with survival data (see Kang and Koehler, 1997; or Cai et al., 2000). We begin by recalling the definition of the strong mixing property. For this we introduce the following notations. Let $\mathcal{F}_i^k(V)$ denote the σ-algebra generated by $\left\{V_j, i \leq j \leq k\right\}$.

Definition 1. Let $\{V_i, i = 1, 2, \ldots\}$ be a strictly stationary sequence of random variables. Given a positive integer n, set

$$\alpha(n) = \sup \left\{ |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{F}_1^k(V) \text{ and } B \in \mathcal{F}_{k+n}^\infty(V),\ k \in \mathbb{N}^* \right\}.$$

The sequence $\{V_i, i \geq 1\}$ is said to be α-mixing if the mixing coefficient $\alpha(n) \rightarrow 0$ as $n \rightarrow \infty$.

This condition was introduced by Rosenblatt (1956). The strong mixing condition is reasonably weak and has many practical applications (for more details, see Doukhan, 1994; Dedecker et al., 2007).

In this paper we are interested in establishing the asymptotic normality of the local linear estimator of the conditional density when the response variable is subject to random censoring and the observations are generated by an *α-mixing* process. In the case of finite dimensional data, Fan et al. (1996) established the joint asymptotic normality of the estimators of the conditional density and its derivative under stationary *ρ-mixing* processes. Liang and Baek (2016) also get similar results under left-truncated and *α-mixing* data. Recently, Xiong et al. (2018) established the asymptotic normality of the local linear estimator of the conditional density for functional time series data. This work will extend their results to *α-mixing* conditions for functional censored data.

This paper is organised as follows. In Section 2 we introduce the estimators of the conditional density function. In Sections 3 and 4 we give some assumptions and comments. The main results are formulated in Section 5. In Sections 6 and 7 we give the proofs of our results.

2. The model

We assume that there exists a regular version of the conditional probability of Y given X, which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and has a twice continuously differentiable density, denoted by $f^x(y)$. When the data are complete, local polynomial smoothing is based on the assumption that the functional parameter is smooth enough to be locally well approximated by a polynomial. In functional statistics, there are several ways for extending the local linear ideas (cf. Baíllo and Grané, 2009; Barrientos-Marin et al., 2010). Here we adopt the fast functional local modelling, that is, we estimate the conditional density $f^x(y)$ by $\widehat{a}$ where the pair $(\widehat{a}, \widehat{b})$ is obtained by minimising the following quantity:

$$\min_{(a,b)\in\mathbb{R}^2} \sum_{i=1}^{n} \left(h_H^{-1} H(h_H^{-1}(y - Y_i)) - a - b\beta(X_i, x) \right)^2 K(h_K^{-1}\delta(x, X_i)),$$

where $\beta(\cdot,\cdot)$ and $\delta(\cdot,\cdot)$ are locating functions defined from $\mathcal{F}^2$ to $\mathbb{R}$, such that $\forall x \in \mathcal{F}$, $\beta(x, x) = 0$ and $d(\cdot,\cdot) = |\delta(\cdot,\cdot)|$, with K and H being kernels and $h_K = h_{K,n}$ (resp. $h_H = h_{H,n}$) chosen as a sequence of positive real numbers.

In the censored case, we adapt the idea of Carbonez et al. (1995), Kohler et al. (2002), and Khardani et al. (2010) to the infinite dimensional case by using a smooth distribution function $H(\cdot)$ instead of a step function. In practice $\overline{G}(\cdot) := 1 - G(\cdot)$ is unknown. Therefore, we replace $\overline{G}(\cdot)$ by its Kaplan–Meier (Kaplan and Meier, 1958) estimate $\overline{G}_n(\cdot)$ given by

$$\overline{G}_n(y) := 1 - G_n(y) = \begin{cases} \prod_{i=1}^{n} \left(1 - \dfrac{1 - \delta_{(i)}}{n - i + 1}\right)^{\mathbf{1}_{\{T_{(i)} \le y\}}} & \text{if } y < T_{(n)}, \\ 0 & \text{otherwise,} \end{cases}$$

where $T_{(1)} < T_{(2)} < \ldots < T_{(n)}$ are the order statistics of T_i and $\delta_{(i)}$ is the concomitant of $T_{(i)}$, which is known to be uniformly convergent to $\overline{G}$. Then the estimator of $f^x(y)$ is defined as $\widehat{a}$ where the pair

$(\widehat{a},\widehat{b})$ minimises the following quantity:

$$\min_{(a,b)\in\mathbb{R}^2}\sum_{i=1}^{n}\left(h_H^{-1}\delta_i\overline{G}_n^{-1}(T_i)H(h_H^{-1}(y-T_i))-a-b\beta(X_i,x)\right)^2K(h_K^{-1}\delta(x,X_i)). \qquad (1)$$

Here we denote $\widehat{a}$ by $\widehat{f^x}(y)$. In addition, the estimator $\widehat{b}$ may similarly be used as an estimator of the *derivative* of $f^x(y)$.

In what follows, we put, for any $x\in\mathcal{F}$, and for all $i=1,\ldots,n$,

$$K_i=K(h_K^{-1}\delta(x,X_i)),\ \beta_i=\beta(X_i,x)\quad\text{and}\quad H_i=H(h_H^{-1}(y-T_i)).$$

Let

$$\mathbf{X}=\begin{pmatrix}1 & \beta_1\\ \vdots & \vdots\\ 1 & \beta_n\end{pmatrix},\qquad \mathbf{H}^*=\begin{pmatrix}h_H^{-1}\delta_1\overline{G}_n^{-1}(T_1)H_1\\ \vdots\\ h_H^{-1}\delta_n\overline{G}_n^{-1}(T_n)H_n\end{pmatrix},\qquad \mathbf{W}=diag(K(h_K^{-1}\delta(x,X_i))).$$

Then, from (1), simple algebra shows that

$$(\widehat{a},\widehat{b})^t=(\mathbf{X}^t\mathbf{W}\mathbf{X})^{-1}\mathbf{X}^t\mathbf{W}\mathbf{H}^*.$$

Let

$$\mathbf{S}_n=\begin{pmatrix}s_{n0} & s_{n1}\\ s_{n1} & s_{n2}\end{pmatrix}\qquad\text{and}\qquad \mathbf{t}_n=\begin{pmatrix}t_{n0}\\ t_{n1}\end{pmatrix},$$

where

$$s_{nj}=\frac{1}{n\mathbb{E}[K_1]}\sum_{i=1}^{n}\left(\frac{\beta_i}{h_K}\right)^jK_i\qquad\text{and}\qquad t_{nj}=\frac{1}{n\mathbb{E}[K_1]}\sum_{i=1}^{n}\left(\frac{\beta_i}{h_K}\right)^jK_ih_H^{-1}\delta_i\overline{G}_n^{-1}(T_i)H_i.$$

Then

$$(\widehat{f^x}(y),\widehat{b})^t=diag(1,h_K^{-1})\mathbf{S}_n^{-1}\mathbf{t}_n.$$

Clearly, by simple algebra, we get explicitly the following expression for $\widehat{f^x}(y)$:

$$\widehat{f^x}(y)=\frac{\sum_{i,j=1}^{n}\delta_i\overline{G}_n^{-1}(T_i)W_{ij}(x)H(h_H^{-1}(y-T_i))}{h_H\sum_{i,j=1}^{n}W_{ij}(x)},$$

where

$$W_{ij}(x)=\beta(X_i,x)\left(\beta(X_i,x)-\beta(X_j,x)\right)K(h_K^{-1}\delta(x,X_i))K(h_K^{-1}\delta(x,X_j)),$$

with the convention $0/0=0$.

3. Notations and hypotheses

In what follows, for any distribution function L, let $\tau_L:=\sup\{t:L(t)<1\}$ be its support's right endpoint. Observe that $\tau_J=\min(\tau_F,\tau_G)$ and consider $\tau<\tau_J$. Let Ω be a compact subset of $(-\infty,\tau]$.

In the sequel, let $C, C_1,$ and C_2 denote generic finite positive constants whose values are unimportant and may change from line to line. Set

$$\begin{aligned}\phi_x(r_1,r_2) &= \mathbb{P}(r_2 \leq \delta(x,X) \leq r_1),\\ \psi_l(\cdot) &= \frac{\partial^l f^{(\cdot)}(y)}{\partial y^l},\\ \Psi_l(s) &= \mathbb{E}\left[\psi_l(X) - \psi_l(x)|\delta(x,X) = s\right], \text{ for some } l \in \{0,2\}.\end{aligned}$$

We will assume the following hypotheses:

(H1) For any $r > 0$, $\phi_x(r) := \phi_x(-r,r) > 0$ and there exists a function $\chi_x(\cdot)$ such that

$$\forall t \in [-1,1], \lim_{h\to 0} \frac{\phi_x(-h,th)}{\phi_x(h)} = \chi_x(t).$$

(H2) For any $l \in \{0,2\}$, the quantities $\Psi_l'(0)$ and $\Psi_l''(0)$ exist, where Ψ_l' and Ψ_l'' denote the first and the second derivatives of Ψ_l, respectively.

(H3) $(Y_i, X_i)_{i\geq 1}$ is a sequence of stationary α-*mixing* rvs with coefficient $\alpha(n)$.

(H4) The mixing coefficient $\alpha(n)$ satisfies:

(i) $\alpha(n) = O(n^{-\lambda})$ for some $\lambda > 3$;

(ii) there exist positive integers $q := q_n$ such that $q = o((n\, h_H \phi_x(h_K))^{1/2})$ and

$$\lim_{n\to\infty} (n\,(h_H\phi_x(h_K))^{-1})^{1/2}\alpha(q) = 0.$$

(H5) The locating operator $\beta(\cdot,\cdot)$ satisfies the following three conditions:

(i) $\forall z \in \mathcal{F},\ C_1\,|\delta(x,z)| \leq |\beta(x,z)| \leq C_2\,|\delta(x,z)|$, where $0 < C_1 < C_2$,

(ii) $\sup_{u\in B(x,r)} |\beta(u,x) - \delta(x,u)| = o(r)$, and

(iii) $h_K \int_{B(x,h_K)} \beta(u,x)dP_X(u) = o(\int_{B(x,h_K)} \beta^2(u,x)dP_X(u))$,

where $B(x,r) = \{z \in \mathcal{F}/|\delta(x,z)| \leq r\}$ is a ball centered at x with radius r and $P_X(u)$ is the probability distribution of X.

(H6) (i) K is a positive, differentiable function supported within $[-1,1]$. Its derivative K' satisfies $K'(t) < 0$, for $-1 \leq t < 1$, and $K(1) > 0$.

(ii) The random variable $\delta(X,x)$ is measurable with respect to the σ-field generated by the random variable $\beta(X,x)$.

(H7) H is a positive function, integrable, bounded, symmetric and such that

$$\int H(t)dt = 1 \quad \text{and} \quad \int t^2 H(t)dt < \infty.$$

(H8) The bandwidths h_K and h_H satisfy

$$\lim_{n\to\infty} h_K = 0, \quad \lim_{n\to\infty} h_H = 0 \quad \text{and} \quad \lim_{n\to\infty} n\, h_H\phi_x(h_K) = \infty.$$

(H9) (i) $\sup_{i\neq j} P[(X_i, X_j) \in B(x, h_K) \times B(x, h_K)] \leq f(x)g(h_K)$ as $h_K \to 0$, where $g(h_K) \to 0$ as $h_K \to 0$ and $f(x)$ is a nonnegative functional in $x \in \mathcal{F}$. We assume that the ratio $g(h_K)/\phi_x^2(h_K)$ is bounded.

(ii) For all $j > 1$, the joint conditional density $f^{(\cdot,\cdot)}(\cdot,\cdot)$ of (Y_1, Y_j) given (X_1, X_j) exists on $\mathbb{R}\times\mathbb{R}\times\mathcal{F}\times\mathcal{F}$ and satisfies $f^{(x_1,x_j)}(y_1, y_j) \leq C$ for $(y_1, y_j, x_1, x_j) \in \mathbb{R}\times\mathbb{R}\times B(x; r_0)\times B(x; r_0)$, where $r_0 > 0$.

(iii) For all $j > 1$, the conditional density $f^{(\cdot,\cdot)}(\cdot)$ of Y_1 given (X_1, X_j) exists on $\mathbb{R}\times\mathcal{F}\times\mathcal{F}$ and satisfies $f^{(x_1,x_j)}(y_1) \leq C$ for $(y_1, x_1, x_j) \in \mathbb{R}\times B(x; r_0)\times B(x; r_0)$.

(iv) For all $j > 1$, the conditional density $f^{(\cdot,\cdot)}(\cdot)$ of Y_j given (X_1, X_j) exists on $\mathbb{R}\times\mathcal{F}\times\mathcal{F}$ and satisfies $f^{(x_1,x_j)}(y_j) \leq C$ for $(y_j, x_1, x_j) \in \mathbb{R}\times B(x; r_0)\times B(x; r_0)$.

(H10) $\lim_{n\to\infty} n\, h_H^5 \phi_x(h_K) = 0$ and $\lim_{n\to\infty} n\, h_H h_K^4 \phi_x(h_K) = 0$.

Remark 1. Assumption (H1) characterises the concentration property of the probability measure of the functional variable X, while Assumption (H2) is a regularity condition which characterises the functional space of our model. Assumptions (H3) and (H4)(i) specify the model and the rate of mixing coefficient. Conditions in (H4) allow us to employ Bernstein's big-block and small-block technique to prove asymptotic normality for an α*-mixing* sequence. Assumption (H5)(i) is necessary to control the shape of the local functional object β; Assumption (H5)(ii) is unrestrictive and Assumption (H5)(iii) is a pivotal hypothesis on the local performance of the operator β. Assumption (H6)(i), (H7) and (H8) are used commonly in the literature. Assumption (H9) is mainly technical, which is employed to simplify the calculations of covariances in the proof. Finally, Assumption (H10) is used to remove the bias term.

4. Main results: asymptotic normality

To give the main result, we list some notations. In the sequel, we let

$$\boldsymbol{M}_a = K^a(1) - \int_{-1}^{1} (K^a(u))' \chi_x(u) du,$$

where $a > 0$,

$$\boldsymbol{N}(a,b) = K^a(1) - \int_{-1}^{1} (u^b K^a(u))' \chi_x(u) du,$$

for all $a > 0$ and $b > 1$, and

$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\boldsymbol{N}(1,2)}{\boldsymbol{M}_1} \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} \frac{\boldsymbol{M}_2}{\boldsymbol{M}_1^2} & 0 \\ 0 & \frac{\boldsymbol{N}(2,2)}{\boldsymbol{M}_1^2} \end{pmatrix}, \qquad \mathbf{U} = \begin{pmatrix} \frac{\boldsymbol{N}(1,2)}{\boldsymbol{M}_1} \\ \frac{\boldsymbol{N}(1,3)}{\boldsymbol{M}_1} \end{pmatrix}.$$

Theorem 1. *Suppose that Assumptions (H1)–(H9) hold. Then*

$$(nh_H\phi_x(h_K))^{1/2}\left[diag(1, h_K)\begin{pmatrix} \hat{f}^x(y) - f^x(y) \\ \widehat{b} - \Psi_0'(0) \end{pmatrix} - \frac{h_K^2}{2}\Psi_0''(0)\mathbf{S}^{-1}\mathbf{U} \right.$$
$$\left. -\frac{h_H^2}{2}\psi_2(x)\int t^2 H(t)dt \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \xrightarrow{D} N\left(0, \overline{G}^{-1}(y) f^x(y) \int H^2(t)dt\mathbf{S}^{-1}\mathbf{V}\mathbf{S}^{-1}\right).$$

Corollary 1. *Suppose that Assumptions (H1)–(H9) hold. Then*

$$(nh_H\phi_x(h_K))^{1/2}\left(\hat{f}^x(y) - f^x(y) - \frac{h_K^2}{2}\Psi_0''(0)\frac{\boldsymbol{N}(1,2)}{\boldsymbol{M}_1} - \frac{h_H^2}{2}\psi_2(x)\int t^2H(t)dt\right) \xrightarrow{D} N(0, V_{HK}^x(y)),$$

where

$$V_{HK}^x(y) = \overline{G}^{-1}(y)f^x(y)\frac{\boldsymbol{M}_2}{\boldsymbol{M}_1^2}\int H^2(t)dt.$$

Now, to construct confidence intervals for $f^x(y)$ we need to remove the bias term and obtain a plug-in estimator of

$$\overline{G}^{-1}(y)\frac{f^x(y)}{h_H}\frac{\boldsymbol{M}_2}{\boldsymbol{M}_1^2 n\phi_x(h_K)}\int H^2(t)dt. \tag{2}$$

Corollary 2. *Suppose that Assumptions (H1)–(H10) hold. Then*

$$\left(\frac{nh_H\phi_x(h_K)}{V_{HK}^x(y)}\right)^{1/2}(\hat{f}^x(y) - f^x(y)) \xrightarrow{D} N(0,1).$$

On the other hand, by Assumptions (H1), (H5) and (H6)(i) we know that $\boldsymbol{M}_2/\boldsymbol{M}_1^2 n\phi_x(h_K)$ can be estimated by $\mathbb{E}(K_1^2)/\mathbb{E}^2(K_1)$, and by applying the kernel estimator of $f^x(y)$ and the Kaplan–Meier estimator of $\overline{G}^{-1}(y)$ given above, the quantity (2) can be estimated by

$$\overline{G}_n^{-1}(y)\frac{\hat{f}^x(y)}{h_H}\frac{\mathbb{E}(K_1^2)}{\mathbb{E}^2(K_1)}\int H^2(t)dt \doteqdot \hat{\sigma}^2(y/x).$$

Then, we approximate a $(1-\gamma)$ confidence interval of $f^x(y)$ by

$$\left[\hat{f}^x(y) - u_{1-\gamma/2}\hat{\sigma}(y/x); \hat{f}^x(y) + u_{1-\gamma/2}\hat{\sigma}(y/x)\right],$$

where $u_{1-\gamma/2}$ denotes the $(1-\gamma/2)$-level quantile of the standard normal distribution.

5. Computational studies

In this section, a simulation study is carried out to investigate the finite-sample performance of the local linear estimator $f_{LL}(y|x)$ of the conditional density function under right-censored and functional dependent data. As everyone knows, the applicability of the asymptotic normality result requires a practical estimation of the asymptotic bias and variance. For this we neglect the bias term and we use a plug-in approach to construct an estimator of the asymptotic variance of the conditional density function given by

$$\overline{G}^{-1}(y)\frac{f^x(y)}{h_H}\frac{\boldsymbol{M}_2}{\boldsymbol{M}_1^2 n\phi_x(h_K)}\int H^2(t)dt. \tag{3}$$

To test the effectiveness of the asymptotic normality result and to gauge its usefulness, let us consider the following regression model where the response is a scalar: $Y_i = r(X_i) + \epsilon_i$, $i = 1, \ldots, n$, where ϵ_i is the error generated by an autoregressive model defined by

$$\epsilon_i = \frac{1}{\sqrt{2}}(\epsilon_{i-1} + \eta_i), \qquad i = 1, \ldots, n,$$

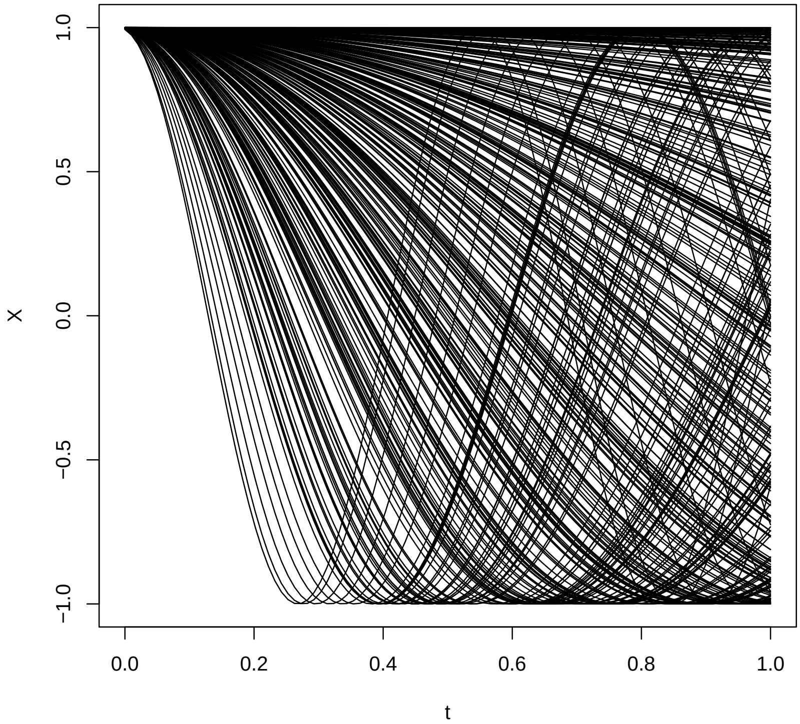

Figure 1. A set of 300 simulated curves.

with $\{\eta_i\}_i$ a sequence of i.i.d. normally distributed random variables with a variance equal to 0.1. The explanatory variables are constructed according to

$$X(t) = A(2 - \cos(\pi t W)) + (1 - A)\cos(\pi t W), \qquad t \in [0, 1],$$

where W is generated from a standard normal distribution and A is a Bernoulli random variable with parameter $p = 0.5$. The X_i are generated from 300 curves and are plotted in Figure 1. On the other hand, n i.i.d. random variables $\{C_i\}_i$ are generated from the exponential distribution $\mathcal{E}(\lambda)$ and for $i = 1, \ldots, n = 300$, the scalar response Y_i is computed by considering the following operator:

$$r(X) = 4\exp\left\{\frac{1}{2 + \int_0^{\pi/2} |X_i(t)|^2}\right\}.$$

Given $X = x$, we can easily see that Y has as a normal distribution with mean $r(x)$ and variance 0.2. Then, we can get the corresponding conditional density, which is explicitly defined by

$$f^{Y|X}(y|x) = \frac{1}{\sqrt{2\pi \times 0.3}} \exp\left\{-\frac{1}{2 \times 0.3}(y - r(x))^2\right\}.$$

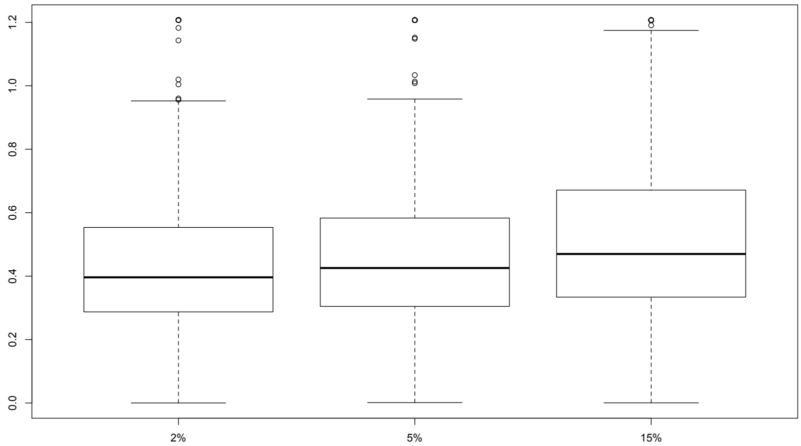

Figure 2. Distribution of the MSE obtained for different CR for $n = 300$.

Therefore, the conditional mode, the conditional mean $r(x)$, and the conditional median functions will coincide and will be equal to $r(x)$, for any fixed x. Our goal, now, involves evaluating the accuracy of the conditional mode function estimator based on randomly censored data. The computation of this estimator is based on the observed data $(X_i, T_i, \delta_i)_{i=1,\ldots,n}$, where $T_i = \min(Y_i, C_i)$ and $\delta_i = \mathbf{1}_{\{Y_i \leq C_i\}}$.

In this simulation study, we present results only for the case where $i = 2$ and $q = 1$. For this, we take $K_0(s) = 3(1 - s^2)\mathbf{1}_{[1,1)}$, $K(1) > 0$, and $K_1(s) = 3(1 - s^2)\mathbf{1}_{[1,1]}$. Elsewhere, as it is well known in FDA, the choice of the metric and the smoothing parameters have crucial roles in computational respects. To optimise these choices for this illustration, we use the local cross-validation procedure method for choosing smoothing parameters h_K and h_H (see Laksaci et al., 2013).

Another important point for ensuring good behaviour of the considered methods is to use locating functions δ and/or β that are well adapted to the kind of data that we have to deal with. Here, it is clear that the shape of the curves (cf. Figure 1) allows us to use the locating functions σ and β defined by the derivatives of the curves. More precisely, we take

$$\delta(x, x') = \left(\int_0^1 (x^{(i)}(t) - x'^{(i)}(t))^2 dt \right)^{1/2} \quad \text{and} \quad \beta(x, x') = \int_0^1 \alpha(t)(x_i(t) - x_i'(t))^2 dt,$$

where $x^{(i)}$ denotes the ith derivative of the curve x, and α is the eigenfunction of the empirical covariance operator $n^{-1} \sum_{i=1}^{n} (X_j^{(i)} - \bar{X}^{(i)})(X_j^{(i)} - \bar{X}^{(i)})$ associated with the $q-$greatest eigenvalues. The performance of the conditional mode estimator $\widehat{\theta}_n(x)$ is evaluated on $N = 400$ replications using different sample sizes $n = 50, 100, 200, 300$. The mean squared error (MSE) is considered here; in particular, for a fixed x, $MSE = \frac{1}{400} \sum_{m=1}^{400} (\widehat{\theta}_{n,m}(x) - r(x))^2$. Figure 2 displays the distribution of the obtained MSE given by the N replications. One can observe that the proposed estimator performs well, especially when the sample size increases. This conclusion is confirmed by Table 1 which provides a numerical summary of the distribution of the MSE with different censored rates (CR).

In the second part of the simulation studies, we are interested in the evaluation of the prediction

Table 1. Numerical summary of the distribution of the MSE, for $N = 400$, obtained for $n = 50, 100, 200$ and 300.

	$CR = 2\%$	$CR = 7\%$	$CR = 16\%$	$CR = 49\%$
$n = 50$	0.392	0.472	0.673	2.431
$n = 100$	0.301	0.332	0.472	1.964
$n = 200$	0.220	0.292	0.346	1. 632
$n = 300$	0.102	0.150	0.215	1.214

accuracy of the conditional median with different censored rates (CR). A sample $(X_i, Y_i)_{i=1,\dots,550}$ of size $n = 550$ generated from the model described above, is considered for this purpose. We split this sample in two parts: a learning subsample $\{(X_i, Y_i); i = 1, \dots, 500\}$, which is used to calculate the predictor (the conditional mode in this case), and a testing subsample $\{(X_i, Y_i); i = 501, \dots, 550\}$, used to evaluate the performance of the predictor. The prediction accuracy is measured for different values of CR by using the Mean Absolute Error (MAE) defined as $MAE = \frac{1}{50}\sum_{i=501}^{550} |Y_i - \widehat{\theta}_n(X_i)|$, as well as the Mean Squared Error (MSE) defined as $MSE = \frac{1}{50}\sum_{i=501}^{550}(Y_i - \widehat{\theta}_n(X_i))^2$. We can see that the prediction accuracy of the conditional mode decreases as the censored rate increases. For censoring distributions we considered $\mathcal{E}(2) - 1, (CR = 7\%, MAE = 0.204, MSE = 0.483)$, $\mathcal{E}(2), (CR = 16\%, MAE = 0.363, MSE = 0.62)$, and $\mathcal{E}(2) + 5, (CR = 49\%, MAE = 1.052, MSE = 2.291)$.

6. Appendix

To prove the main result we need the following lemmas.

Lemma 1 (see Rachdi et al., 2014)**.** *Suppose that Assumptions (H1), (H5) and (H6)(i) hold. Then,*

(a) $\mathbb{E}[K_1^a] = \boldsymbol{M}_a \phi_x(h_k) + o(\phi_x(h_k))$, *for* $a > 0$*;*

(b) $\mathbb{E}[K_1^a \beta_1] = o(h_k \phi_x(h_k))$, *for all* $a > 0$*;*

(c) $\mathbb{E}[K_1^a \beta_1^b] = \boldsymbol{N}(a,b) h_k^b \phi_x(h_k) + o(h_k^b \phi_x(h_k))$, *for all* $a > 0, b > 1$.

Lemma 2. *Suppose that Assumptions (H2), (H5)–(H7) and (H9) hold. Then,*

(a) $h_H^{-2}\mathbb{E}[K_1 K_j \delta_1 \overline{G}^{-1}(T_1) H_1 \delta_j \overline{G}^{-1}(T_j) H_j] = O(g(h_K))$*;*
$h_H^{-1}\mathbb{E}[K_1 K_j \delta_1 \overline{G}^{-1}(T_1) H_1] = O(g(h_K))$*;*
$h_H^{-1}\mathbb{E}[K_1 K_j \delta_j \overline{G}^{-1}(T_j) H_j] = O(g(h_K))\mathbb{E}[K_1 K_j] = O((g(h_K))$, *for all* $j > 1$.

(b)
$$h_H^{-1}\mathbb{E}[K_1^a \beta_1^b \delta_1 \overline{G}^{-c}(T_1) H_1^c] = \overline{G}^{1-c}(y) \int H^c(t)dt \left\{\psi_0(x)\mathbb{E}[K_1^a \beta_1^b] + \Psi_0'(0)\mathbb{E}[K_1^a \beta_1^{b+1}]\right\}$$
$$+ \frac{h_H^2}{2}\overline{G}^{1-c}(y) \int t^2 H^c(t)dt \left\{\psi_2(x)\mathbb{E}[K_1^a \beta_1^b] + \Psi_2'(0)\mathbb{E}[K_1^a \beta_1^{b+1}]\right\}$$
$$+ o(\mathbb{E}[K_1^a \beta_1^{b+1}]) + o(h_H^2 \mathbb{E}[K_1^a \beta_1^b])$$
for $a > 0$, $b = 0$ *and* $c = 0$, *or* $b > 1$ *and* $c > 1$.

Lemma 3 (see Xiong et al., 2018)**.** *Suppose that Assumptions (H1)–(H9) hold. Then*

$$s_{n0} \xrightarrow{P} 1, \quad s_{n1} \xrightarrow{P} 0, \quad s_{n2} \xrightarrow{P} \frac{\boldsymbol{N}(1,2)}{\boldsymbol{M}_1}, \quad s_{n3} \xrightarrow{P} \frac{\boldsymbol{N}(1,3)}{\boldsymbol{M}_1}.$$

Lemma 4 (see Volkonskii and Rozanov, 1959)**.** *Let $V_1,...,V_m$ be α-mixing random variables measurable with respect to the σ-algebras $\mathcal{F}_{i_1}^{j_1},...,\mathcal{F}_{i_m}^{j_m}$, respectively, with $1 \le i_1 < j_1 < ... < j_m \le n$, $1 \le w \le i_{l+1} - j_l$ and $|V_j| \le 1$ for $l, j = 1,2,...,m$. Then,*

$$\left| \mathbb{E}\left[\prod_{j=1}^{m} V_j\right] - \prod_{j=1}^{m} \mathbb{E}[V_j] \right| \le 16(m-1)\alpha_w,$$

where $\mathcal{F}_a^b = \sigma\{V_i, a \le i \le b\}$ and α_w is the mixing coefficient.

Lemma 5 (see Davydov, 1968, Corollary, p. 692)**.** *Suppose that X and Y are random variables satisfying $\mathbb{E}|X|^p < \infty$, $\mathbb{E}|Y|^q < \infty$, where $p, q > 1$, $p^{-1} + q^{-1} < 1$. Then*

$$|\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]| \le 8\mathbb{E}^{p^{-1}}|X|^p\mathbb{E}^{q^{-1}}|Y|^q \left\{ \sup_{A\in\sigma(x),B\in\sigma(Y)} |p(A\cap B) - p(A)p(B)| \right\}^{1-p^{-1}-q^{-1}}.$$

Proof of Theorem 1

Define

$$T_{nj} = \frac{1}{n\mathbb{E}[K_1]}\sum_{i=1}^{n}\left(\frac{\beta_i}{h_K}\right)^j K_i(h_H^{-1}\delta_i\overline{G}_n^{-1}(T_i)H_i - \mathbb{E}[f^{X_i}(y)|\beta_i]),$$

$$t_{nj}^* = \frac{1}{n\mathbb{E}[K_1]}\sum_{i=1}^{n}\left(\frac{\beta_i}{h_K}\right)^j K_i(h_H^{-1}\delta_i\overline{G}^{-1}(T_i)H_i - \mathbb{E}[f^{X_i}(y)|\beta_i]),$$

and let $\mathbf{T}_n = (T_{n0}, T_{n1})^\tau$ and $\mathbf{t}_n^* = (t_{n0}^*, t_{n1}^*)^\tau$. In view of (H2), when $|\beta_i| \le Ch_K$, by Taylor expansion, we get

$$\Psi_0(\beta_i) = \Psi_0'(0)\beta_i + \frac{1}{2}\Psi_0''(0)\beta_i^2 + o(\beta_i^2).$$

Then,

$$\begin{aligned} \mathbb{E}[f^{X_i}(y)|\beta_i] &= f^x(y) + \mathbb{E}[f^{X_i}(y) - f^x(y)|\beta_i] \\ &= f^x(y) + \Psi_0(\beta_i) \\ &= f^x(y) + \Psi_0'(0)\beta_i + \frac{1}{2}\Psi_0''(0)\beta_i^2 + o(\beta_i^2). \end{aligned} \tag{4}$$

Therefore,

$$(\mathbb{E}[f^{X_1}(y)|\beta_1], ..., \mathbb{E}[f^{X_n}(y))|\beta_n])^\tau = \mathbf{X}\begin{pmatrix} f^x(y) \\ \Psi_0'(0) \end{pmatrix} + \frac{1}{2}\Psi_0''(0)(\beta_1^2, ..., \beta_n^2)^\tau + (o(\beta_1^2), ..., o(\beta_n^2))^\tau.$$

Then

$$\begin{aligned} \mathbf{S}_n^{-1}\mathbf{t}_n^* = diag(1, h_K)\begin{pmatrix} \hat{f}^x(y) - f^x(y) \\ \hat{\Psi}_0'(0) - \Psi_0'(0) \end{pmatrix} - \frac{h_K^2}{2}\Psi_0''(0)\mathbf{S}_n^{-1}\begin{pmatrix} s_{n2} \\ s_{n3} \end{pmatrix} \\ - o(h_K^2)\mathbf{S}_n^{-1}\begin{pmatrix} s_{n2} \\ s_{n3} \end{pmatrix} - \mathbf{S}_n^{-1}(\mathbf{T}_n - \mathbf{t}_n^*). \end{aligned}$$

Lemma 3 implies that

$$\mathbf{S}_n \xrightarrow{P} \mathbf{S},$$

and

$$\mathbf{S}_n^{-1}\begin{pmatrix} s_{n2} \\ s_{n3} \end{pmatrix} \xrightarrow{P} \mathbf{S}^{-1}\mathbf{U}.$$

The rest of the proof is divided into the following two steps:
Step 1. We verify that

$$\mathbf{S}_n^{-1}(\mathbf{T}_n - \mathbf{t}_n^*) = O_p\left(\sqrt{\frac{\log(\log n)}{n}}\right).$$

Note that, for $0 \le j \le 1$, we have

$$\begin{aligned}|T_{nj} - t_{nj}^*| &= \left|\frac{1}{n\mathbb{E}[K_1]}\sum_{i=1}^{n}\left(\frac{\beta_i}{h_K}\right)^j K_i h_H^{-1}\delta_i H_i\left[\overline{G}_n^{-1}(T_i) - \overline{G}^{-1}(T_i)\right]\right| \\ &\le \sup_{t\in\Omega}\left|\overline{G}_n^{-1}(t) - \overline{G}^{-1}(t)\right| \frac{1}{nh_K^j\mathbb{E}[K_1]}\sum_{i=1}^{n}\left|\beta_i^j K_i h_H^{-1} H_i\right|.\end{aligned}$$

Then from $(nh_K^j\mathbb{E}[K_1])^{-1}\sum_{i=1}^n|\beta_i^j K_i h_H^{-1}H_i| = O_p(1)$, Lemma 3 and the LIL on the censoring law (see formula (4.28) in Deheuvels and Einmahl, 2000), one obtains the result.
Step 2. We prove that

$$(nh_H\phi_x(h_K))^{1/2}\left\{t_n^* - \frac{h_H^2}{2}\psi_2(x)\int t^2H(t)dt\begin{pmatrix}1\\0\end{pmatrix} + o(h_H^2)\right\} \xrightarrow{D} N(0,\overline{G}^{-1}(y)f^x(y)\int H^2dt\mathbf{V}). \tag{5}$$

For any given vector of real numbers $\mathrm{a} = (a_0, a_1)^\tau \ne 0$, set

$$U_i = \frac{(h_H\phi_x(h_K))^{1/2}}{\mathbb{E}[K_1]}\left(a_0 + a_1\frac{\beta_i}{h_K}\right)K_i\left(h_H^{-1}\delta_i\overline{G}^{-1}(T_i)H_i - \mathbb{E}[f^{X_i}(y)|\beta_i]\right), 1 \le i \le n.$$

Then

$$(nh_H\phi_x(h_K))^{1/2}\mathrm{a}^\tau t_n^* = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(U_i - \mathbb{E}[U_i]) + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{E}[U_i]. \tag{6}$$

From the Cramér–Wold theorem and Equation (6), Equation (5) will hold if we can prove the following two claims:

Claim 1. $\frac{1}{\sqrt{n}}\sum_{i=1}^n\mathbb{E}[U_i] = (nh_H\phi_x(h_K))^{1/2}\left\{\frac{a_0h_H^2\psi_2(x)}{2}\int t^2H(t)dt + o(h_H^2)\right\}.$

Claim 2. $\frac{1}{\sqrt{n}}\sum_{i=1}^n(U_i - \mathbb{E}[U_i]) \xrightarrow{D} N(0, \Delta^2(y|x)), \Delta^2(y|x) := \overline{G}^{-1}(y)f^x(y)\int H^2(t)dt\mathrm{a}^\tau\mathbf{V}\mathrm{a}.$

Proof of Claim 1.

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{E}[U_i] &= \sqrt{n}\mathbb{E}[U_1]\\ &= \frac{a_0(nh_H\phi_x(h_K))^{1/2}}{\mathbb{E}[K_1]}\{h_H^{-1}\mathbb{E}[K_1\delta_1\overline{G}^{-1}(T_1)H_1] - \mathbb{E}[K_1\psi_0(X_1)]\}\\ &\quad + \frac{a_1(nh_H\phi_x(h_K))^{1/2}}{h_K\mathbb{E}[K_1]}\{h_H^{-1}\mathbb{E}[K_1\beta_1\delta_1\overline{G}^{-1}(T_1)H_1] - \mathbb{E}[K_1\beta_1\psi_0(X_1)]\}.\end{aligned}$$

For $a = c = 1$, we obtain from Lemma 2 that

$$\begin{aligned}&h_H^{-1}\mathbb{E}[K_1\beta_1^b\delta_1\overline{G}^{-1}(T_1)H_1] - \mathbb{E}[K_1\beta_1^b\psi_0(X_1)]\\ &= \frac{h_H^2}{2}\int t^2H(t)dt\{\psi_2(x)\mathbb{E}[K_1\beta_1^b] + \Psi_2'(0)\mathbb{E}[K_1\beta_1^{b+1}] + o(\mathbb{E}[K_1\beta_1^{b+1}])\} + o(h_H^2\mathbb{E}[K_1\beta_1^b]).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{E}[U_i] &= \frac{a_0(nh_H\phi_x(h_K))^{1/2}}{\mathbb{E}[K_1]}\mathbb{E}[K_1]\left\{\frac{h_H^2}{2}\int t^2H(t)dt\psi_2(x) + o(h_H^2)\right\}\\ &\quad + \frac{a_1(nh_H\phi_x(h_K))^{1/2}}{h_K\mathbb{E}[K_1]}\left\{\frac{h_H^2}{2}\int t^2H(t)dt\{\mathbb{E}[K_1\beta_1]\psi_2(x) + \mathbb{E}[K_1\beta_1^2]\Psi_2'(0)\right.\\ &\quad \left. + o(\mathbb{E}[K_1\beta_1^2])\} + o(h_H^2\mathbb{E}[K_1\beta_1])\right\}.\end{aligned}$$

Claim 1 now follows from Lemma 1.

Proof of Claim 2. The proof is similar to the proof of Theorem 4.1 in Xiong et al. (2018).

Assumption (H4)(ii) implies that there is a sequence of positive integers $\delta_n \to \infty$, such that $\delta_n q_n = o((nh_H\phi_x(h_K))^{1/2})$ and $\delta_n(n(h_H\phi_x(h_K))^{-1})^{1/2}\alpha(q_n) \to 0$. Let $\omega = [\frac{n}{r+q}]$ and $r = [(nh_H\phi_x(h_K))^{1/2}/\delta_n]$. Then,

$$q/r \to 0, \omega\alpha(q) \to 0, \omega q/n \to 0, r/n \to 0, r/(nh_H\phi_x(h_K))^{1/2} \to 0. \tag{7}$$

Next we will employ Bernstein's big-block and small-block procedure. Partition the set $\{1, 2, \ldots.., n\}$ into $2\omega + 1$ subsets with large block size r and small block size q. Let

$$W_i = U_i - \mathbb{E}[U_i], 1 \le i \le n, \zeta_{mn} = \sum_{i=k_m}^{k_m+r-1} W_i, \zeta'_{mn} = \sum_{i=l_m}^{l_m+q-1} W_i, \zeta''_{mn} = \sum_{i=\omega(r+q)+1}^{n} W_i,$$

where $k_m = (m-1)(r+q) + 1$, $l_m = (m-1)(r+q) + r + 1$, $m = 1, \ldots.., \omega$. Then,

$$\begin{aligned}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(U_i - \mathbb{E}[U_i]) = n^{-1/2}\sum_{i=1}^{n}W_i &= n^{-1/2}\left\{\sum_{m=1}^{\omega}\zeta_{mn} + \sum_{m=1}^{\omega}\zeta'_{mn} + \zeta''_{\omega n}\right\}\\ &:= n^{-1/2}\{\mathfrak{I}_{1n} + \mathfrak{I}_{2n} + \mathfrak{I}_{3n}\}.\end{aligned}$$

Then it suffices to show that

$$n^{-1}\mathbb{E}(\mathfrak{I}_{2n})^2 \to 0, n^{-1}\mathbb{E}(\mathfrak{I}_{3n})^2 \to 0, Var[n^{-1/2}\mathfrak{I}_{1n})] \to \Delta^2(y|x), \tag{8}$$

$$\left|\mathbb{E}[\exp(it\sum_{m=1}^{\omega} n^{-1/2}\zeta_{mn})] - \prod_{m=1}^{\omega}\mathbb{E}[\exp(itn^{-1/2}\zeta_{mn})]\right| \to 0, \tag{9}$$

$$A_n(\varepsilon) = \frac{1}{n}\sum_{m=1}^{\omega}\mathbb{E}[\zeta_{mn}^2 I(|\zeta_{mn}| > \varepsilon\Delta(y|x)\sqrt{n})] \to 0, \forall\varepsilon > 0. \tag{10}$$

We first prove (8). Write

$$\begin{aligned} n^{-1}\mathbb{E}(\mathfrak{I}_{2n})^2 &= \frac{1}{n}\sum_{m=1}^{\omega}\sum_{i=l_m}^{l_m+q-1}\mathbb{E}[W_i^2] + \frac{2}{n}\sum_{m=1}^{\omega}\sum_{l_m\le i<j\le l_m+q-1} Cov(W_i, W_j) + \frac{2}{n}\sum_{1\le i<j\le\omega} Cov(\zeta'_{in}, \zeta'_{jn}) \\ &:= J_{1n} + J_{2n} + J_{3n}. \end{aligned}$$

Calculate $\mathbb{E}[W_1^2] = \mathbb{E}[U_1^2] - \mathbb{E}^2[U_1]$, where we note that $\mathbb{E}[U_1^2]$ is equal to

$$\begin{aligned} &\frac{h_H\phi_x(h_K)}{\mathbb{E}^2[K_1]}\mathbb{E}[(a_0 + a_1\frac{\beta_1}{h_K})^2 K_1^2(h_H^{-1}\delta_1\overline{G}^{-1}(T_1)H_1 - \mathbb{E}[f^{X_1}(y)|\beta_1])^2] \\ = &\frac{a_0^2 h_H\phi_x(h_K)}{\mathbb{E}^2[K_1]}\{h_H^{-2}\mathbb{E}[K_1^2\delta_1\overline{G}^{-2}(T_1)H_1^2] - 2h_H^{-1}\mathbb{E}[K_1^2\delta_1\overline{G}^{-1}(T_1)H_1\mathbb{E}[f^{X_1}(y)|\beta_1]] \\ &+ \mathbb{E}[K_1^2\mathbb{E}^2[f^{X_1}(y)|\beta_1]]\} \\ &+ \frac{a_1^2 h_H\phi_x(h_K)}{h_K^2\mathbb{E}^2[K_1]}\{h_H^{-2}\mathbb{E}[K_1^2\beta_1^2\delta_1\overline{G}^{-2}(T_1)H_1^2] - 2h_H^{-1}\mathbb{E}[K_1^2\beta_1^2\delta_1\overline{G}^{-1}(T_1)H_1\mathbb{E}[f^{X_1}(y)|\beta_1]] \\ &+ \mathbb{E}[K_1^2\beta_1^2\mathbb{E}^2[f^{X_1}(y)|\beta_1]]\} \\ &+ \frac{2a_0a_1h_H\phi_x(h_K))}{h_K\mathbb{E}^2[K_1]}\{h_H^{-2}\mathbb{E}[K_1^2\beta_1\delta_1\overline{G}^{-2}(T_1)H_1^2] - 2h_H^{-1}\mathbb{E}[K_1^2\beta_1\delta_1\overline{G}^{-1}(T_1)H_1\mathbb{E}[f^{X_1}(y)|\beta_1]] \\ &+ \mathbb{E}[K_1^2\beta_1\mathbb{E}^2[f^{X_1}(y)|\beta_1]]\}. \end{aligned}$$

Equation (4), Assumptions (H6)(i), (H5) and (H2) imply that

$$K_1^2\mathbb{E}[f^{X_1}(y)|\beta_1] = K_1^2 O(1) \text{ and } K_1^2\mathbb{E}^2[f^{X_1}(y)|\beta_1] = K_1^2 O(1).$$

Then we get

$$\begin{aligned} \mathbb{E}[U_1^2] =&\frac{a_0^2 h_H\phi_x(h_K)}{\mathbb{E}^2[K_1]}\{h_H^{-2}\mathbb{E}[K_1^2\delta_1\overline{G}^{-2}(T_1)H_1^2] - 2h_H^{-1}\mathbb{E}[K_1^2\delta_1\overline{G}^{-1}(T_1)H_1]O(1) + \mathbb{E}[K_1^2]O(1)\} \\ &+ \frac{a_1^2 h_H\phi_x(h_K)}{h_K^2\mathbb{E}^2[K_1]}\{h_H^{-2}\mathbb{E}[K_1^2\beta_1^2\delta_1\overline{G}^{-2}(T_1)H_1^2] \\ &- 2h_H^{-1}\mathbb{E}[K_1^2\beta_1^2\delta_1\overline{G}^{-1}(T_1)H_1]O(1) + \mathbb{E}[K_1^2\beta_1^2]O(1)\} \\ &+ \frac{2a_0a_1h_H\phi_x(h_K))}{h_K\mathbb{E}^2[K_1]}\{h_H^{-2}\mathbb{E}[K_1^2\beta_1\delta_1\overline{G}^{-2}(T_1)H_1^2] \\ &- 2h_H^{-1}\mathbb{E}[K_1^2\beta_1\delta_1\overline{G}^{-1}(T_1)H_1]O(1) + \mathbb{E}[K_1^2\beta_1]O(1)\}, \end{aligned}$$

which, along with Lemma 1 and Lemma 2(b), implies that

$$\begin{aligned}\mathbb{E}[U_1^2] &\to a_0^2\frac{\boldsymbol{M}_2}{\boldsymbol{M}_1^2}\overline{G}^{-1}(y)\int H^2(t)dt\psi_0(x)+a_1^2\frac{\boldsymbol{N}(2,2)}{\boldsymbol{M}_1^2}\overline{G}^{-1}(y)\int H^2(t)dt\psi_0(x)\\ &= a_0^2\frac{\boldsymbol{M}_2}{\boldsymbol{M}_1^2}\overline{G}^{-1}(y)\int H^2(t)dtf^x(y)+a_1^2\frac{\boldsymbol{N}(2,2)}{\boldsymbol{M}_1^2}\overline{G}^{-1}(y)\int H^2(t)dtf^x(y)\\ &= \overline{G}^{-1}(y)f^x(y)\int H^2(t)dt\mathrm{a}^\tau\mathbf{V}\mathrm{a}. \end{aligned}\tag{11}$$

From Claim 1 we have $\mathbb{E}[U_1]\to 0$. Hence,

$$\mathbb{E}[W_1^2]=\mathbb{E}[U_1^2]-\mathbb{E}^2[U_1]\to\Delta^2(y|x),\tag{12}$$

which yields $J_{1n}=O(\omega q/n)\to 0$ by Equation (7). From the definition of $\mathfrak{I}_{2n}$, we know

$$|J_{2n}|\le\frac{2}{n}\sum_{1\le i<j\le n}|Cov(W_i,W_j)|=\frac{2}{n}\sum_{1\le i<j\le n}|Cov(U_i,U_j)|,$$

$$|J_{3n}|\le\frac{2}{n}\sum_{1\le i<j\le n}|Cov(W_i,W_j)|=\frac{2}{n}\sum_{1\le i<j\le n}|Cov(U_i,U_j)|.$$

Therefore, to prove $J_{2n}=o(1)$ and $J_{3n}=o(1)$, we need only prove that

$$\frac{1}{n}\sum_{1\le i<j\le n}|Cov(U_i,U_j)|\to 0.\tag{13}$$

Take $c_n=[h_H\phi_x(h_K)]^{-(1-\frac{1}{\lambda})/\eta}$, for some $1-1/\lambda<\eta<\lambda-2$. Then we set

$$G_1=\{(1,j):j\in\{1,......,n\},1\le j-1\le c_n\},$$

$$G_2=\{(1,j):j\in\{1,......,n\},c_n+1\le j-1\le n-1\}.$$

According to the above splitting, we get

$$\frac{1}{n}\sum_{1\le i<j\le n}|Cov(U_i,U_j)|=\sum_{j\in G_1}(1-\frac{j-1}{n})|Cov(U_1,U_j)|+\sum_{j\in G_2}(1-\frac{j-1}{n})|Cov(U_1,U_j)|.\tag{14}$$

Note that

$$|Cov(U_i,U_j)|\le|\mathbb{E}[U_1U_j]|+\mathbb{E}^2[U_1].\tag{15}$$

Equation (4), Assumptions (H6)(i), (H5) and (H2) imply that

$$K_1\mathbb{E}[f^{X_1}(y)|\beta_1]=K_1O(1)\text{ and }\frac{\beta_1}{h_K}K_1=K_1O(1).\tag{16}$$

Then

$$\begin{aligned}
|\mathbb{E}[U_1U_j]| &\leq \mathbb{E}|U_1U_j| \\
&= \frac{h_H\phi_x(h_K)}{\mathbb{E}^2[K_1]}\mathbb{E}\left|\left(a_0 + a_1\frac{\beta_1}{h_K}\right)K_1\left(h_H^{-1}\delta_1\overline{G}^{-1}(T_1)H_1 - \mathbb{E}[f^{X_1}(y)|\beta_1|]\right)\right. \\
&\quad \left.\left(a_0 + a_1\frac{\beta_j}{h_K}\right)K_j\left(h_H^{-1}\delta_j\overline{G}^{-1}(T_j)H_j - \mathbb{E}[f^{X_j}(y)|\beta_j|]\right)\right|. \\
&\leq \frac{h_H\phi_x(h_K)}{\mathbb{E}^2[K_1]}\mathbb{E}[O(1)K_1K_j(h_H^{-1}\delta_1\overline{G}^{-1}(T_1)H_1 + O(1))(h_H^{-1}\delta_j\overline{G}^{-1}(T_j)H_j + O(1))] \\
&= O(1)\frac{h_H\phi_x(h_K)}{\mathbb{E}^2[K_1]}\left\{h_H^{-2}\mathbb{E}[K_1K_j\delta_1\overline{G}^{-1}(T_1)H_1\delta_j\overline{G}^{-1}(T_j)H_j]\right. \\
&\quad \left. + h_H^{-1}\mathbb{E}[K_1K_j\delta_1\overline{G}^{-1}(T_1)H_1] + h_H^{-1}\mathbb{E}[K_1K_j\delta_j\overline{G}^{-1}(T_j)H_j] + \mathbb{E}[K_1K_j]\right\},
\end{aligned}$$

which, with Lemma 2(a) and Assumption (H9)(i), implies that

$$|\mathbb{E}[U_1U_j]| = O(h_H\phi_x(h_K)). \tag{17}$$

In addition, Claim 1 implies that

$$\mathbb{E}^2[U_1] = O(h_H^5\phi_x(h_K)). \tag{18}$$

Then, Equations (15), (17) and (18) yield

$$Cov(U_1, U_j) = O(h_H\phi_x(h_K)). \tag{19}$$

Equation (19) implies that

$$\begin{aligned}
\sum_{j\in G_1}\left(1 - \frac{j-1}{n}\right)|Cov(U_1, U_j)| &= O(1)\sum_{j=1}^{c_n} h_H\phi_x(h_K) = O(c_nh_H\phi_x(h_K)) \\
&= O(1)[h_H\phi_x(h_K)]^{1-(1-\frac{1}{\lambda})/\eta} \to 0.
\end{aligned} \tag{20}$$

On the other hand, it follows from Lemma 5 that

$$|Cov(U_1, U_j)| \leq 8[\mathbb{E}|U_1|^{2\lambda}]^{1/\lambda}[\alpha(j-1)]^{1-\frac{1}{\lambda}}. \tag{21}$$

The C^r-inequality and Equation (16) imply that

$$\begin{aligned}
\mathbb{E}|[U_1]|^{2\lambda} &= \frac{(h_H\phi_x(h_K))^\lambda}{\mathbb{E}^{2\lambda}[K_1]}\mathbb{E}\left|\left(a_0 + a_1\frac{\beta_1}{h_K}\right)K_1(h_H^{-1}\delta_1\overline{G}^{-1}(T_1)H_1 - \mathbb{E}[f^{X_1}(y)|\beta_1])\right|^{2\lambda} \\
&= \frac{(h_H\phi_x(h_K))^\lambda}{\mathbb{E}^{2\lambda}[K_1]}\mathbb{E}\left|O(1)K_1(h_H^{-1}\delta_1\overline{G}^{-1}(T_1)H_1 + O(1))\right|^{2\lambda} \\
&\leq O(1)\frac{(h_H\phi_x(h_K))^\lambda}{\mathbb{E}^{2\lambda}[K_1]}\{\mathbb{E}[h_H^{-2\lambda}K_1^{2\lambda}\delta_1\overline{G}^{-2\lambda}(T_1)H_1^{2\lambda}] + \mathbb{E}[K_1^{2\lambda}]\} \\
&= O(1)\frac{(h_H\phi_x(h_K))^\lambda}{\mathbb{E}^{2\lambda}[K_1]}\{h_H^{1-2\lambda}\mathbb{E}[h_H^{-1}K_1^{2\lambda}\delta_1\overline{G}^{-2\lambda}(T_1)H_1^{2\lambda}] + \mathbb{E}[K_1^{2\lambda}]\},
\end{aligned}$$

again by Lemma 1 and Lemma 2(b),

$$\mathbb{E}|U_1|^{2\lambda} = O((h_H \phi_x(h_K))^{1-\lambda}). \tag{22}$$

It follows from Equations (21) and (22) that

$$\begin{aligned}
\sum_{j\in G_2}(1-\frac{j-1}{n})|Cov(U_1,U_j)| &= O(1)\sum_{j\in G_2}[\alpha(j-1)]^{1-\frac{1}{\lambda}}(h_H\phi_x(h_K))^{-(1-\frac{1}{\lambda})} \\
&\le O(1)\sum_{m\ge C_n+1} m^{-(\lambda-1)}(h_H\phi_x(h_K))^{-(1-\frac{1}{\lambda})} \\
&\le O(1)C_n^{-(\lambda-2)}(h_H\phi_x(h_K))^{-(1-\frac{1}{\lambda})} \\
&\le O(1)(h_H\phi_x(h_K))^{-(1-\frac{1}{\lambda})}[h_H\phi_x(h_K)]^{\frac{(1-\frac{1}{\lambda})(\lambda-2)}{\eta}} \to 0.
\end{aligned} \tag{23}$$

Equations (14), (20) and (23) imply (13). Then, we have $n^{-1}E(\Im_{2n})^2 \to 0$. As to $n^{-1}E(\Im_{3n})^2$, by (7), (11) and (13),

$$\begin{aligned}
n^{-1}\mathbb{E}(\Im_{3n})^2 &\le \frac{1}{n}\sum_{i=\omega(r+q)+1}^{n}\mathbb{E}[W_i^2] + \frac{2}{n}\sum_{1\le i<j\le n}^{n}|Cov(W_i,W_j)| \\
&\le \frac{1}{n}\sum_{i=\omega(r+q)+1}^{n}\mathbb{E}[U_i^2] + \frac{2}{n}\sum_{1\le i<j\le n}^{n}|Cov(U_i,U_j)| \\
&\le C.\frac{n-\omega(r+q)}{n} + \frac{2}{n}\sum_{1\le i<j\le n}^{n}|Cov(U_i,U_j)| \to 0.
\end{aligned}$$

Besides, since $\omega r/n \to 1$, it follows from (12) and (13) that

$$\begin{aligned}
Var[n^{-1/2}\Im_{1n}] &= \frac{1}{n}\mathbb{E}(\Im_{1n})^2 \\
&= \frac{1}{n}\sum_{m=1}^{\omega}\sum_{i=k_m}^{k_m+r-1}\mathbb{E}[W_i^2] + \frac{2}{n}\sum_{m=1}^{\omega}\sum_{k_m\le i<j\le k_m+r-1}|Cov(W_i,W_j)| \\
&\quad + \frac{2}{n}\sum_{1i<j\le n}|Cov(\zeta_{in},\zeta_{jn})| \\
&= \frac{\omega r}{n}\mathbb{E}[W_i]^2 + O\left(\frac{1}{n}\sum_{1\le i<j\le n}|Cov(U_i,U_j)|\right) \to \Delta^2(y|x),
\end{aligned}$$

and so (8) is proved.

As to (9), according to (7) and Lemma 4, we have

$$\left|\mathbb{E}[\exp(it\sum_{m=1}^{\omega}n^{-1/2}\zeta_{mn})] - \prod_{m=1}^{\omega}\mathbb{E}[\exp(itn^{-1/2}\zeta_{mn})]\right| \le 16(\omega-1)\alpha(q+1) \le 16\omega\alpha(q) \to 0.$$

Finally, we establish (10). Thinking about $\max_{1\le m\le\omega}|\zeta_{mn}|$, from (16), Assumptions (H6)(i), (H7) and Lemma 1(a), we have

$$
\begin{aligned}
&\frac{(h_H\phi_x(h_K))^{1/2}}{r}\max_{1\le m\le\omega}|\zeta_{mn}| \\
&\le \frac{(h_H\phi_x(h_K))^{1/2}}{r}\max_{1\le m\le\omega}\sum_{i=k_m}^{k_m+r-1}|W_i| \\
&\le \frac{(h_H\phi_x(h_K))^{1/2}}{r}\max_{1\le m\le\omega}\sum_{i=k_m}^{k_m+r-1}\left|\frac{(h_H\phi_x(h_K))^{1/2}}{\mathbb{E}[K_1]}\right| \\
&\times[|(a_0+a_1\frac{\beta_i}{h_K})K_i(h_H^{-1}\delta_i\overline{G}^{-1}(T_i)H_i-\mathbb{E}[f^{X_i}(y)|\beta_i])| \\
&+|\mathbb{E}[(a_0+a_1\frac{\beta_i}{h_K})K_i(h_H^{-1}\delta_i\overline{G}^{-1}(T_i)H_i-\mathbb{E}[f^{X_i}(y)|\beta_i])]|] \\
&= \frac{(h_H\phi_x(h_K))^{1/2}}{r}\max_{1\le m\le\omega}\sum_{i=k_m}^{k_m+r-1}\frac{(h_H\phi_x(h_K))^{1/2}}{\mathbb{E}[K_1]}[O(1)K_i(h_H^{-1}\delta_i\overline{G}^{-1}(T_i)H_i+O(1)) \\
&+O(1)\mathbb{E}[K_i(h_H^{-1}\delta_i\overline{G}^{-1}(T_i)H_i+O(1))]] \\
&= \frac{h_H\phi_x(h_K)}{r}\max_{1\le m\le\omega}\sum_{i=k_m}^{k_m+r-1}O(h_H^{-1}) \\
&= O(1).
\end{aligned}
$$

Therefore, we can get

$$\max_{1\le m\le\omega}|\zeta_{mn}| = O\left(r/\sqrt{h_H\phi_x(h_K)}\right),$$

which means that for large n,

$$I(|\zeta_{mn}|>\varepsilon\Delta(y|x)\sqrt{n})=0,$$

by the fact that $r/(h_H\phi_x(h_K))^{1/2}\to 0$ in (7). Therefore, $A_n(\varepsilon)\to 0$. ■

Proof of Lemma 2

(a) Under Assumptions (H2), (H5)–(H7) and (H9) and by a simple calculations we can show the results.
(b) Take conditional expectation,

$$h_H^{-1}\mathbb{E}[K_1^a\beta_1^b\delta_1\overline{G}^{-c}(T_1)H_1^c] = h_H^{-1}\mathbb{E}[K_1^a\beta_1^b\mathbb{E}[\delta_1\overline{G}^{-c}(T_1)H_1^c|X_1]].$$

According to $\mathbb{E}[\delta_1|Y_1]=\overline{G}(Y_1)$, we find

$$
\begin{aligned}
\mathbb{E}[\delta_1\overline{G}^{-c}(T_1)H_1^c|X_1] &= \mathbb{E}[\mathbb{E}[\delta_1\overline{G}^{-c}(T_1)H_1^c|Y_1]|X_1]] \\
&= \mathbb{E}[\overline{G}^{-c}(Y_1)H_1^c\mathbb{E}[\delta_1|Y_1]|X_1]] \\
&= \mathbb{E}[\overline{G}^{1-c}(Y_1)H_1^c|X_1],
\end{aligned}
$$

which yield that

$$\begin{aligned} h_H^{-1}\mathbb{E}[K_1^a\beta_1^b\delta_1\overline{G}^{-c}(T_1)H_1^c] &= h_H^{-1}\mathbb{E}[K_1^a\beta_1^b)\mathbb{E}[\overline{G}^{1-c}(Y_1)H_1^c|X_1]] \\ &= h_H^{-1}\mathbb{E}[K_1^a\beta_1^b\int\overline{G}^{1-c}(s)H^c(\frac{y-s}{h_H})f^{X_1}(s)ds] \\ &= \mathbb{E}[K_1^a\beta_1^b\int\overline{G}^{1-c}(y-h_Ht)H^c(t)f^{X_1}(y-h_Ht)dt]. \end{aligned} \tag{24}$$

Applying the Taylor expansion of order 2 to $f^{X_1}(\cdot)$ in y, we obtain

$$\begin{aligned} f^{X_1}(y-h_Ht) &= f^{X_1}(y) - h_Ht\frac{\partial f^{X_1}(y)}{\partial y} + \frac{h_H^2t^2}{2}\frac{\partial^2 f^{X_1}(y)}{\partial^2 y} + o(h_H^2t^2) \\ &= \psi_0(X_1) - h_Ht\psi_1(X_1) + \frac{h_H^2t^2}{2}\psi_2(X_1) + o(h_H^2t^2), \end{aligned}$$

which, combined with (24), Assumption (H7), and from the fact that $G(\cdot)$ is continuous, yields

$$\begin{aligned} h_H^{-1}\mathbb{E}[K_1^a\beta_1^b\delta_1\overline{G}^{-c}(T_1)H_1^c] &= \overline{G}^{1-c}(y)\int H^c(t)dt\mathbb{E}[K_1^a\beta_1^b\psi_0(X_1)] \\ &+ \frac{h_H^2}{2}\overline{G}^{1-c}(y)\int t^2H^c(t)dt\mathbb{E}[K_1^a\beta_1^b\psi_2(X_1)] + o(h_H^2\mathbb{E}[K_1^a\beta_1^b]). \end{aligned} \tag{25}$$

Now, following Rachdi et al. (2014), we show that

$$\begin{aligned} \mathbb{E}[K_1^a\beta_1^b\psi_l(X_1)] &= \psi_l(x)\mathbb{E}[K_1^a\beta_1^b] + \mathbb{E}[K_1^a\beta_1^b(\psi_l(X_1)-\psi_l(x))] \\ &= \psi_l(x)\mathbb{E}[K_1^a\beta_1^b] + \mathbb{E}[K_1^a\beta_1^b\mathbb{E}[\psi_l(X_1)-\psi_l(x)|\beta_1]] \\ &= \psi_l(x)\mathbb{E}[K_1^a\beta_1^b] + \mathbb{E}[K_1^a\beta_1^b\Psi_l(\beta_1)], \end{aligned}$$

and since $\Psi_l(0) = 0$ for $l \in \{0,2\}$ and Assumption (H2), by the Taylor expansion of order 1, we obtain

$$\mathbb{E}[K_1^a\beta_1^b\Psi_l(\beta_1)] = \Psi_l'(0)\mathbb{E}[K_1^a\beta_1^{b+1}] + o(\mathbb{E}[K_1^a\beta_1^{b+1}]),$$

which, together with (25), implies

$$\begin{aligned} h_H^{-1}\mathbb{E}[K_1^a\beta_1^b\delta_1\overline{G}^{-c}(T_1)H_1^c] &= \overline{G}^{1-c}(y)\int H^c(t)dt\left\{\psi_0(x)\mathbb{E}[K_1^a\beta_1^b] + \Psi_0'(0)\mathbb{E}[K_1^a\beta_1^{b+1}]\right\} \\ &+ \frac{h_H^2}{2}\overline{G}^{1-c}(y)\int t^2H^c(t)dt\left\{\psi_2(x)\mathbb{E}[K_1^a\beta_1^b] + \Psi_2'(0)\mathbb{E}[K_1^a\beta_1^{b+1}]\right\} \\ &+ o(\mathbb{E}[K_1^a\beta_1^{b+1}]) + o(h_H^2\mathbb{E}[K_1^a\beta_1^b]). \end{aligned}$$

■

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