BAYES ESTIMATION OF LORENZ CURVE AND GINI-INDEX FOR POWER FUNCTION DISTRIBUTION

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Summary: In this article, we estimate the shape parameter, Lorenz curve and Gini-index for power function distributions using a Bayesian method. Bayes estimators have been developed under squared error loss function as well as under weighted squared error loss function. We demonstrate the use of the proposed estimation procedure with the U. S. average income data for the period 1913-2010. Our proposed Bayesian estimators are compared using a Monte Carlo simulation study with the ML estimators proposed by Belzunce, Candel and Ruiz (1998).

1. Introduction

The Lorenz curve is a graphical representation, usually adopted to depict the distribution of income and wealth in a population. Let X be a continuous non-negative random variable representing income of a society or community with distribution function $F(x)$, Gastwirth (1971) defined the Lorenz curve corresponding to X as

$$L(p) = \frac{1}{E(X)} \int_0^p Q(u)\,\mathrm{du}, \qquad 0 \le p \le 1,\ E(X) < \infty, \tag{1}$$

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where $Q(u)$ is the quantile function. Clearly $L(p)$ gives the fraction of total income that the holders of the lowest p^{th} fraction of income possesses. Most of the measures of income inequality are derived from the Lorenz curve. An important measure of inequality is the Gini-index associated with $L(p)$ and is defined as

$$G = 1 - 2\int_0^1 L(p)\mathrm{dp}. \tag{2}$$

This is a ratio of the area between the Lorenz curve and the 45° line to the area under the 45° line. In general, these notions are useful for measuring concentration and inequality in the distributions of resources and in size distributions. For the applications of Lorenz curve and Gini-index we refer to Moothathu (1985), Moothathu (1990) and the references therein. These measures have also been found applications in reliability theory. For more details, see Chandra and Singpurwalla (1981), Sathar, Suresh and Nair (2007) and Sathar and Nair (2009).

Moothathu (1991) has derived uniformly minimum variance unbiased estimators of the Lorenz curve and Gini-index for lognormal and Pareto distributions respectively. Sathar, Jeevanand and Nair (2005), Sathar and Suresh (2006) and Sathar and Jeevanand (2009) have discussed the Bayesian estimation of the Lorenz curve and Gini-index of the Pareto and exponential distributions respectively. For recent works on the estimation of the Lorenz curve and Gini-index, we refer to Hasegawa and Kozumi (2003), Rohde (2009), Sarabia, Prieto and Sarabia (2010), Fellman (2012) and the references therein.

The present article is organized as follows. In Section 2, we consider the maximum likelihood estimates of the shape parameter, Lorenz curve and Gini-index of the power distribution. We also discussed the bias-corrected maximum likelihood estimates in Section 2. Section 3 deals with the Bayesian estimation of the shape parameter, Lorenz curve and Gini-index of the power distribution for the case when both scale and shape parameters of the distribution are unknown. In Section 4, we demonstrate the use of the proposed estimation procedure with the U. S. income data for the period 1913-2010. Based on a Monte Carlo simulation study, comparisons are made between the proposed estimators, ML estimators and bias-corrected MLEs based on the bias and mean squared errors. These comparisons are presented in Section 5. We utilize Section 6 for some concluding remarks and for the description of the summary of the results developed in this work.

2. The Model and ML Estimates

Among the models which provide a better fit to the whole income distribution, there are the Singh-Maddala model and the Dagum Model Type-I (Dagum, 1980). Belzunce et al. (1998) observed that for low values of the parameters of the Singh-Maddala distribution, the right residual income follows, asymptotically, the power function distribution. Therefore in the study of poverty, it is important to consider the estimation of the Lorenz curve and the Gini-index for this model.

Let $\{X_i\}$, $i = 1, 2, ..., n$ be a sequence of independent and identically distributed random variables from a power function distribution with pdf

$$f(x|\beta,\alpha) = \frac{\alpha}{\beta}\left(\frac{\beta}{x}\right)^{-(\alpha-1)}, \quad 0 < x < \beta, \quad \alpha,\beta > 0, \tag{3}$$

where β and α are scale and shape parameters, respectively. The Lorenz curve and the Gini-index for (3) can be simplified respectively as

$$L(p) = p^{1+\alpha^{-1}}, \quad 0 < p < 1, \tag{4}$$

and

$$G = (1+2\alpha)^{-1}. \tag{5}$$

Bagchi and Sarkar (1986) discussed the Bayes interval estimation for the shape parameter of the power distribution. For recent works on estimation of the parameters of the power function distribution, we refer to Sinha, Singh, Singh and Singh (2008), Sultan, Sultan and Ahmad (2014) and the references therein.

Belzunce et al. (1998) obtained the ML estimates of the parameters α, the Lorenz curve $L(p)$ and the Gini-index G and are given respectively as

$$\widehat{\alpha}_{ML} = \frac{1}{\widehat{\lambda}_1}, \ \widehat{L(p)}_{ML} = p^{1+\widehat{\lambda}_1} \ \text{and} \ \widehat{G}_{ML} = \widehat{\lambda}_2, \tag{6}$$

where

$$\widehat{\lambda}_1 = S, \ \widehat{\lambda}_2 = \frac{S}{2+S}, \ S = \frac{1}{n}\sum_{j=1}^{n}\left(Y_j - Y_{(1)}\right), \ Y_{(1)} = \min(Y_i) \ \text{and} \ Y_i = -\ln X_i.$$

2.1. Bias - Corrected Maximum Likelihood Estimation

A bias adjusted MLE can be constructed by subtracting the bias (estimated at the MLE's of the parameters) from the original MLE. For some arbitrary distribution, let $l(\theta)$ be the likelihood based on a sample of n observations, with p-dimensional parameter vector, θ. The log-likelihood $L = \ln l(\theta)$ is assumed to be regular with respect to all derivatives up to and including the third order. Cordeiro and Klein (1994) show that the expression for the bias of the MLE of θ ($\widehat{\theta}$) can be rewritten in the convenient matrix form as

$$Bias\left(\widehat{\theta}\right) = K^{-1}A \ \text{vec}\left(K^{-1}\right) + O\left(n^{-2}\right). \tag{7}$$

The terms in (7) are defined in Giles, Feng and Godwin (2011) as,

$$K = \left\{-k_{ij}\right\}, i, j = 1, 2, \ldots, p,$$

and

$$A = \left[A^{(1)}|A^{(2)}|\cdots|A^{(p)}\right],$$

respectively. The elements of the matrices K and A are defined in Giles et al. (2011) as,

$$A^{(l)} = \left\{a_{ij}^{(l)}\right\}, i, j, l = 1, 2, \ldots, p$$
$$a_{ij}^{(l)} = k_{ij}^{(l)} - \frac{k_{ijl}}{2}, i, j, l = 1, 2, \ldots, p$$
$$k_{ij}^{(l)} = \frac{\partial k_{ij}}{\partial \theta_l}, i, j, l = 1, 2, \ldots, p$$
$$k_{ijl} = E\left(\frac{\partial^3 L}{\partial \theta_i \partial \theta_j \partial \theta_l}\right), i, j, l = 1, 2, \ldots, p$$
$$k_{ij} = E\left(\frac{\partial^2 L}{\partial \theta_i \partial \theta_j}\right), i, j = 1, 2, \ldots, p$$

A bias-corrected MLE for θ can then be obtained as,

$$\widetilde{\theta} = \widehat{\theta} - \widehat{Bias}\left(\widehat{\theta}\right). \tag{8}$$

For more details, see Cox and Snell (1968), Firth (1993), Giles et al. (2011) and the references therein. The asymptotic expansions of bias for the general OLS regression models is given in Gabaix and Ibragimov (2011).

For the power function distribution (3), the bias for the shape parameter α is obtained as

$$\widehat{Bias}\left(\widehat{\alpha}\right) = \frac{\alpha\left(3 + 2\alpha\right)}{2n\left(1 + \alpha\right)^2}, \tag{9}$$

and the bias-corrected MLE can be obtained as

$$\widetilde{\alpha}_{ML} = \widehat{\alpha}_{ML} - \frac{\widehat{\alpha}_{ML}\left(3 + 2\widehat{\alpha}_{ML}\right)}{2n\left(1 + \widehat{\alpha}_{ML}\right)^2}. \tag{10}$$

The bias-corrected MLE's for the Lorenz curve and Gini-index can be given as

$$\widetilde{L(p)}_{ML} = p^{1+\left(\widetilde{\alpha}_{ML}\right)^{-1}} \tag{11}$$

and

$$\widetilde{G}_{ML} = \left(1 + 2\widetilde{\alpha}_{ML}\right)^{-1} \tag{12}$$

respectively.

3. Bayesian Estimation

Recently, the Bayesian approach has received much attention for analyzing statistical data and has been often proposed as a valid alternative to traditional statistical perspectives. The Bayesian approach allows prior subjective knowledge on parameters to be incorporated into the inferential procedure. Hence, Bayesian methods usually require less sample data to achieve the same quality of inferences than methods based on sampling theory, which becomes extremely important in case of expensive testing procedures.

Bayesian statistics provide a conceptually simple process for updating uncertainty in the light of evidence. From a decision-theoretic view point, in order to select the 'best' estimator, a loss function must be specified and is used to represent a penalty associated with each of the possible estimates. Nonetheless, it has been observed that in certain situations when one loss is the true loss function, the Bayes estimate under another loss function performs better than the Bayes estimate under the true loss. Therefore, we consider symmetric as well as asymmetric loss functions for getting better understanding in our Bayesian analysis.

The use of squared error loss function (SELF) is justified when the loss is symmetric in nature. It is also popular because of its mathematical simplicity. The Bayes estimator of ϕ, denoted by $\widehat{\phi}_{BS}$ under SELF is the posterior mean of ϕ and is given by

$$\widehat{\phi}_{BS} = E_{\phi}(\phi|\underline{x}). \tag{13}$$

But the nature of losses are not always symmetric and hence we also used an asymmetric loss function, weighted squared error loss function (WSELF). Under WSELF, the Bayes estimator of ϕ, denoted by $\widehat{\phi}_{BW}$ is given by

$$\widehat{\phi}_{BW} = \left[E_{\phi}\left(\phi^{-1}|\underline{x}\right)\right]^{-1} \tag{14}$$

provided that the expectation $E_{\phi}\left(\phi^{-1}|\underline{x}\right)$ exists and is finite.

3.1. Estimation when α and β are unknown

The most general and perhaps a more realistic situation is when both the shape and scale parameters of the distribution are unknown. In this section, we consider the problem of estimation of α, $L(p)$ and G when α and β are unknown. In Bayesian inference, a prior probability distribution, often called simply the prior, of an uncertain parameter θ or latent variable is a probability distribution that expresses uncertainty about θ before the data are taken into account. The parameters of a prior distribution are called hyperparameters, to distinguish them from the parameters (Θ) of the model. The Bayesian deduction requires appropriate choice of priors for the parameters. Arnold and Press (1983) pointed out that, from a strict Bayesian viewpoint, there is clearly no way in which one can say that one prior is better than any other. An individual chooses his or her subjective prior and must then contend with its advantages and disadvantages. But if we have enough information about the parameter(s) then it is better to make use of the informative prior(s) which may certainly be preferred over all other choices.

The likelihood function corresponding to this set-up can be written as

$$l(\underline{x}|\alpha,\beta) = \alpha^{n}\beta^{-n\alpha}\exp[(\alpha-1)z], \tag{15}$$

where

$$z = \sum_{i=1}^{n} \ln x_i.$$

Here, we suggest the joint prior distribution for the parameters α and β as

$$g(\alpha,\beta) = g(\beta|\alpha)\, g(\alpha), \tag{16}$$

where

$$g(\beta|\alpha) = \beta^{-1},$$

which is the Jeffrey's prior and choosing the gamma prior for α as

$$g(\alpha) = \frac{\tau^r}{\Gamma(r)} \alpha^{r-1} \exp(-\tau\alpha), \;\; r, \tau, \alpha > 0. \tag{17}$$

The gamma prior is one of the priors most often considered by researchers due to its mathematical simplicity. Also, the gamma prior belongs to the conjugate prior family of distributions. The joint posterior density can be simplified as

$$f(\alpha, \beta|\underline{x}) = \frac{\tau^r}{\Gamma(r)} \alpha^{n+r-1} \beta^{-n\alpha-1} \exp(-\alpha\tau) \exp[(\alpha-1)z], \;\; \alpha > 0, \;\; \beta > X_{(n)}, \tag{18}$$

where $X_{(n)} = \max(X_1, X_2, ..., X_n)$.

Now the marginal posterior density of α is obtained by integrating the joint posterior density (18) with respect to β and is given by

$$f(\alpha|\underline{x}) = \frac{Z^{M-1}}{\Gamma(M-1)} \alpha^{M-2} \exp(-\alpha Z), \tag{19}$$

where $M = n + r$ and $Z = \tau - z + n \ln X_{(n)}$.

3.1.1. Estimators based on Squared Error Loss Function

Assuming the SELF, the Bayes estimator of α is obtained by using the posterior pdf (19) and is obtained as

$$\widehat{\alpha}_{BS} = E(\alpha|\underline{x}) = \frac{M-1}{Z}. \tag{20}$$

The Bayes estimator of Lorenz curve $L(p)$ under SELF is obtained by using (4) and (19) and is obtained as

$$\widehat{L(p)}_{BS} = E(L(p)|\underline{x}) = \frac{pZ^{M-1}}{\Gamma(M-1)} \sum_{t=0}^{\infty} \frac{(\ln p)^t}{t!} \frac{\Gamma(M-t-1)}{Z^{M-t-1}}. \tag{21}$$

This is obtained by making use of the representation in Gradshteyn and Ryzhik (2007) given as

$$a^x = \sum_{k=0}^{\infty} \frac{(x \ln a)^k}{k!}. \tag{22}$$

The Bayes estimator of Gini-index G under SELF is obtained by using (5) and (19) and is obtained as

$$\widehat{G}_{BS} = E(G|\underline{x}) = Z^{M-1} 2^{1-M} \exp\left(\frac{Z}{2}\right) \Gamma\left(2-M, \frac{Z}{2}\right), \tag{23}$$

where the incomplete gamma function $\Gamma\left(2-M, \frac{Z}{2}\right)$ satisfies

$$\Gamma\left(2-M, \frac{Z}{2}\right) = \int_{\frac{Z}{2}}^{\infty} y^{2-M-1} \exp(-y) \, \mathrm{dy}.$$

3.1.2. Estimators based on Weighted Squared Error Loss Function

The Bayes estimator for α under WSELF is

$$\widehat{\alpha}_{BW} = \left[E\left(\alpha^{-1}|\underline{x}\right)\right]^{-1} = \frac{M-2}{Z}. \tag{24}$$

The Bayes estimator for the Lorenz curve $L(p)$ under WSELF is obtained by using (4), (19) and (22) and is obtained as

$$\widehat{L(p)}_{BW} = \left[E\left(L(p)^{-1}|\underline{x}\right)\right]^{-1} = \frac{p\Gamma(M-1)}{Z^{M-1}}\left[\sum_{t=0}^{\infty}\frac{(\ln p)^t}{(-1)^t\,t!}\frac{\Gamma(M-t-1)}{Z^{M-t-1}}\right]^{-1}. \tag{25}$$

The Bayes estimator for the Gini-index G under WSELF is obtained by using (5) and (19) and is obtained as

$$\widehat{G}_{BW} = \left[E\left(G^{-1}|\underline{x}\right)\right]^{-1} = \frac{Z}{Z+2M-2}. \tag{26}$$

4. A numerical example

To illustrate the usefulness of the proposed estimators obtained in Section 3 with real situations, we considered here the real data set reported by Saez (2012) representing the average income (excluding capital gains) in the United States for the period 1913-2010. We fit the power function distribution to the right proportional residual income of this data. For finding the right residual income, we choose the right residual income level as $50,000$. The fit seems to be good. (For reference, the Anderson-Darling statistic equals 2.4762 with a p-value of 0.0511).

For this model, using MLE, the estimated parameters are $\widehat{\alpha}_{ML} = 1.57$ and $\widehat{\beta}_{ML} = 0.9966$. We use the value of $p = 0.5$, for evaluating the estimates of the Lorenz curve. The ML estimators for the Lorenz curve and Gini-index for this data are obtained as 0.3215 and 0.2415 respectively. The bias-corrected ML estimators for the shape parameter α, Lorenz curve $L(p)$ and Gini-index G for this data are obtained as $\widetilde{\alpha}_{ML} = 1.5617$, $\widetilde{L(p)}_{ML} = 0.3208$ and $\widetilde{G}_{ML} = 0.2425$ respectively.

Based on this data, we evaluate and present the proposed estimates of the Lorenz curve and Gini-index in Table 1. For studying the effect of hyperparameters, the estimators are evaluated for the combinations of the values of these hyperparameters given in Table 1. From Table 1, we can see that the estimators are not sensitive to variation in the values of the hyperparameters. It is clear from Table 1 that the performance of the Lorenz curve and Gini-index estimators along with the shape parameter α using SELF and WSELF are more or less similar. The Lorenz curve for different values of p based on this data is depicted in Figure 1. From the Figure 1, it can be seen that the proposed and the ML estimators of Lorenz curve are very close to each other.

Table 1: Estimates of α, $L(p)$ and G for U. S. income data.

$(r,\ \tau)$	SELF			WSELF		
	α	$L(p)$	G	α	$L(p)$	G
(3 , 1)	1.5519	0.3186	0.2453	1.5344	0.3179	0.2437
(4 , 1)	1.5693	0.3202	0.2432	1.5519	0.3195	0.2416
(4 , 2)	1.5424	0.3178	0.2464	1.5253	0.3170	0.2448
(5 , 1)	1.5867	0.3218	0.2411	1.5693	0.3211	0.2396
(5 , 2)	1.5595	0.3194	0.2443	1.5424	0.3186	0.2428
(6 , 1)	1.6042	0.3234	0.2391	1.5867	0.3227	0.2376
(6 , 2)	1.5767	0.3209	0.2423	1.5595	0.3202	0.2408
(6 , 3)	1.5501	0.3185	0.2454	1.5333	0.3178	0.2439
(7 , 2)	1.5938	0.3225	0.2403	1.5767	0.3218	0.2388
(7 , 3)	1.5670	0.3201	0.2434	1.5501	0.3194	0.2419

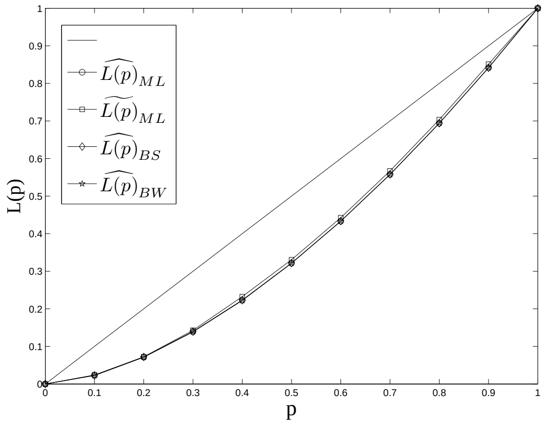

Figure 1: Estimates of Lorenz Curve (U. S. Income data).

5. Monte Carlo simulation

In order to assess the performance of the estimators obtained in Section 3, we present here a simulation study. All the programmes were written using the Mathematica 7 package. The simulation study was done according to the following steps.

Step 1: Generate a sample of size $n = 50$, 5000 and 10000 from the power function distribution (3) with $\alpha = 0.5$, 0.8, 1.0, 1.5 and $\beta = 1.0$.

Step 2: For the vector (r, τ) of hyperparameters, calculate the estimates of α, $L(p)$ and G using (20), (21), (23), (24), (25) and (26) respectively. The ML estimates are computed using (6). The bias-corrected MLE's are computed using (10), (11) and (12).

Step 3: Repeat steps 1 and 2, 1000 times and calculate the mean squared error (MSE) and bias for each estimate. The results are tabulated in Tables 2 - 4.

Table 2: Bias and MSEs (in parenthesis) of the estimates of α.

	α	0.5	0.8	1.0	1.5
$n = 5000$	$\widehat{\alpha}_{ML}$	7.060×10^{-4}	-1.663×10^{-2}	2.256×10^{-3}	3.010×10^{-4}
		(4.901×10^{-5})	(1.525×10^{-3})	(1.762×10^{-4})	(4.049×10^{-4})
	$\widetilde{\alpha}_{ML}$	6.170×10^{-4}	-1.777×10^{-2}	2.130×10^{-3}	1.570×10^{-4}
		(4.888×10^{-5})	(1.529×10^{-3})	(1.756×10^{-4})	(4.048×10^{-4})
	$\widehat{\alpha}_{BS}$	6.104×10^{-4}	-2.008×10^{-3}	1.589×10^{-3}	-1.386×10^{-4}
		(4.876×10^{-5})	(1.533×10^{-4})	(1.725×10^{-4})	(4.041×10^{-4})
	$\widehat{\alpha}_{BW}$	5.103×10^{-4}	-2.167×10^{-3}	1.389×10^{-3}	-1.375×10^{-4}
		(4.863×10^{-5})	(1.539×10^{-4})	(1.719×10^{-4})	(4.042×10^{-4})
$n = 10000$	$\widehat{\alpha}_{ML}$	8.015×10^{-4}	-2.399×10^{-4}	-4.251×10^{-3}	-5.511×10^{-4}
		(2.741×10^{-5})	(6.441×10^{-5})	(1.122×10^{-4})	(2.122×10^{-4})
	$\widetilde{\alpha}_{ML}$	7.570×10^{-4}	-2.967×10^{-4}	-4.173×10^{-3}	-5.231×10^{-4}
		(2.734×10^{-5})	(6.443×10^{-5})	(1.128×10^{-4})	(2.122×10^{-4})
	$\widehat{\alpha}_{BS}$	7.292×10^{-4}	-4.129×10^{-5}	-4.040×10^{-3}	-1.321×10^{-4}
		(2.721×10^{-5})	(6.439×10^{-6})	(1.147×10^{-4})	(2.118×10^{-4})
	$\widehat{\alpha}_{BW}$	7.091×10^{-4}	-4.929×10^{-5}	-4.140×10^{-3}	-1.471×10^{-4}
		(2.733×10^{-5})	(6.445×10^{-6})	(1.155×10^{-4})	(2.122×10^{-4})

Table 3: Bias and MSEs (in parenthesis) of the estimates of $L(p)$.

	α	0.5	0.8	1.0	1.5
	True $L(p)$	0.1250	0.2102	0.2500	0.3150
$n = 5000$	$\widehat{L(p)}_{ML}$	2.345×10^{-4}	-4.030×10^{-4}	3.713×10^{-4}	9.472×10^{-6}
		(5.857×10^{-6})	(8.013×10^{-6})	(5.268×10^{-6})	(3.811×10^{-6})
	$\widetilde{L(p)}_{ML}$	2.036×10^{-4}	-4.289×10^{-4}	3.497×10^{-4}	-4.501×10^{-6}
		(5.844×10^{-6})	(8.036×10^{-6})	(5.253×10^{-6})	(3.812×10^{-6})
	$\widehat{L(p)}_{BS}$	1.907×10^{-4}	-3.021×10^{-4}	2.336×10^{-4}	-1.766×10^{-4}
		(5.824×10^{-6})	(7.083×10^{-6})	(5.159×10^{-6})	(3.631×10^{-6})
	$\widehat{L(p)}_{BW}$	1.427×10^{-4}	-3.337×10^{-4}	2.096×10^{-4}	-1.901×10^{-4}
		(5.810×10^{-6})	(7.119×10^{-6})	(5.150×10^{-6})	(3.637×10^{-6})
$n = 10000$	$\widehat{L(p)}_{ML}$	2.720×10^{-4}	-6.479×10^{-5}	-7.249×10^{-4}	-5.416×10^{-5}
		(3.284×10^{-6})	(3.332×10^{-6})	(3.398×10^{-6})	(2.136×10^{-6})

Continued $\cdots$

	α	0.5	0.8	1.0	1.5
	True $L(p)$	0.1250	0.2102	0.2500	0.3150
	$\widetilde{L(p)}_{ML}$	2.566×10^{-4}	-7.772×10^{-5}	-7.357×10^{-4}	-6.115×10^{-5}
		(3.275×10^{-6})	(3.334×10^{-6})	(3.414×10^{-6})	(2.127×10^{-6})
	$\widehat{L(p)}_{BS}$	2.520×10^{-4}	-1.145×10^{-5}	-6.936×10^{-4}	-1.498×10^{-5}
		(3.261×10^{-6})	(3.325×10^{-6})	(3.293×10^{-6})	(2.020×10^{-6})
	$\widehat{L(p)}_{BW}$	2.280×10^{-4}	-1.303×10^{-5}	-7.057×10^{-4}	-1.565×10^{-5}
		(3.253×10^{-6})	(3.327×10^{-6})	(3.312×10^{-6})	(2.023×10^{-6})

Table 4: Bias and MSEs (in parenthesis) of the estimates of G.

	α	0.5	0.8	1.0	1.5
	True G	0.5000	0.3846	0.3333	0.2500
$n = 5000$	$\widehat{G}_{ML}$	-3.291×10^{-4}	5.264×10^{-4}	-4.757×10^{-4}	-1.283×10^{-4}
		(1.216×10^{-5})	(1.355×10^{-5})	(8.662×10^{-6})	(6.325×10^{-6})
	$\widetilde{G}_{ML}$	-2.847×10^{-4}	5.601×10^{-4}	-4.480×10^{-4}	5.172×10^{-5}
		(1.213×10^{-5})	(1.359×10^{-5})	(8.639×10^{-6})	(6.326×10^{-6})
	$\widehat{G}_{BS}$	-2.564×10^{-4}	4.575×10^{-4}	-2.984×10^{-4}	2.262×10^{-5}
		(1.209×10^{-5})	(1.367×10^{-5})	(6.484×10^{-6})	(4.356×10^{-6})
	$\widehat{G}_{BW}$	-2.614×10^{-4}	4.284×10^{-4}	-3.281×10^{-4}	1.980×10^{-5}
		(1.210×10^{-5})	(1.363×10^{-5})	(6.502×10^{-6})	(4.344×10^{-6})
$n = 10000$	$\widehat{G}_{ML}$	-3.873×10^{-4}	8.532×10^{-4}	9.299×10^{-4}	6.944×10^{-5}
		(6.824×10^{-6})	(5.625×10^{-6})	(5.589×10^{-6})	(3.344×10^{-6})
	$\widetilde{G}_{ML}$	-3.651×10^{-4}	1.021×10^{-4}	9.438×10^{-4}	7.844×10^{-5}
		(6.808×10^{-6})	(5.629×10^{-6})	(5.616×10^{-6})	(3.346×10^{-6})
	$\widehat{G}_{BS}$	-3.537×10^{-4}	1.511×10^{-5}	1.018×10^{-4}	1.923×10^{-5}
		(4.816×10^{-6})	(3.632×10^{-6})	(3.746×10^{-6})	(2.352×10^{-6})
	$\widehat{G}_{BW}$	-3.562×10^{-4}	1.365×10^{-5}	1.004×10^{-4}	1.782×10^{-5}
		(4.826×10^{-6})	(3.628×10^{-6})	(3.716×10^{-6})	(2.346×10^{-6})

The Lorenz curve for different values of p based on the simulated samples from the power function distribution is depicted in Figure 2. From the Figure 2, it can be seen that the Lorenz curve estimated using the proposed estimators are very close to the true value.

6. Conclusion

The present paper proposes Bayesian approaches to estimate α, $L(p)$ and G for a power function distribution. The estimators are obtained using both symmetric and asymmetric loss functions. Comparisons are made between the different estimators based on a simulation study and a practical example using a real data set. The effect of symmetric and asymmetric loss functions was examined and the following were observed:

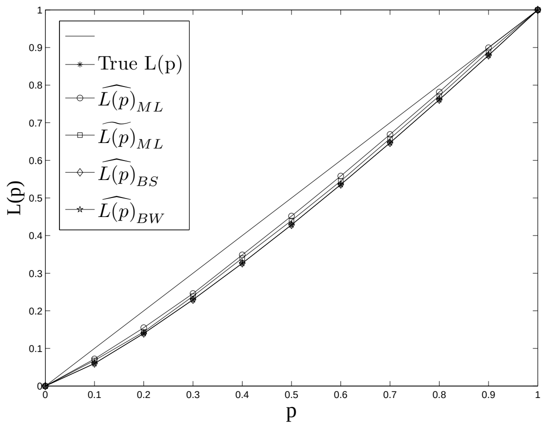

Figure 2: Estimates of Lorenz Curve (Simulated data).

1. From Table 1, we can conclude that the prior hyperparameters r and τ are not sensitive to the estimates of α, $L(p)$ and G.
2. From Tables 2 - 4, the bias and MSEs of the estimates of α, $L(p)$ and G decreases with increase in the sample size.
3. From Tables 2 - 4, we can conclude that the proposed Bayesian estimates of α, $L(p)$ and G show better performance in terms of bias and MSEs than the ML estimates and the bias-corrected ML estimates.
4. From Tables 2 - 4, we can conclude that the bias-corrected MLE's for α, $L(p)$ and G show better performance in terms of bias and MSEs than the ML estimates.

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