SEMI-PARAMETRIC SIMULTANEOUS CONFIDENCE BANDS FOR AN ORDINAL DOMINANCE CURVE

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Abstract: Practitioners in various fields are often confronted with the task of assessing the ability of a diagnostic test to correctly assign one of two labels (healthy/diseased or low/high credit risk) to each observation in a collection of observations. An ordinal dominance curve (ODC) describes the functional relationship between the proportions of correct assignments to the labels. In this paper we derive a semi-parametric confidence band for the ODC and evaluate some of its properties by Monte Carlo simulation. The methodology is illustrated by application to data from a credit risk environment.

1. Introduction

Practitioners are often confronted with the task of assessing the ability of a diagnostic test to accurately assign one of two labels to each observation in a collection of observations. The inherent discriminatory capacity of any diagnostic test depends on the extent to which the probability distributions of the realised outcomes are separated or overlap. There are numerous application areas mentioned in the literature wherein the development of a diagnostic test that correctly classifies its subjects into distinct groups is of utmost importance. For example, the quality of a medical diagnostic test is primarily measured by its ability to distinguish with high precision between healthy and diseased individuals. A relatively more recent application area that is gaining momentum is credit risk assessment. The global financial crisis of 2008 has led to increased scrutiny of the risk profile of potential and existing applicants for loans or credit facilities. In assessing the credit worthiness of a client, scorecards (rating systems) are designed with the specific purpose of distinguishing between low risk and high risk applicants.

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The measurements emanating from a diagnostic test in the two groups are realizations of random variables X and Y which we will assume have continuous and strictly increasing distribution functions F and G respectively. The ordinal dominance curve (ODC), $\varphi(t),\ 0<t<1$, is constructed by plotting $t=G(c)$ on the horizontal axis against $F(c)=FG^{-1}(t)$ on the vertical axis for $-\infty<c<\infty$. Thus,

$$\varphi(t)=F\left(G^{-1}(t)\right)=P(G(X)\le t).$$

A typical *ODC* is displayed in Figure 1. The 45° line depicts a situation in which the diagnostic test has no predictive value i.e. classification is equivalent to flipping an unbiased coin.

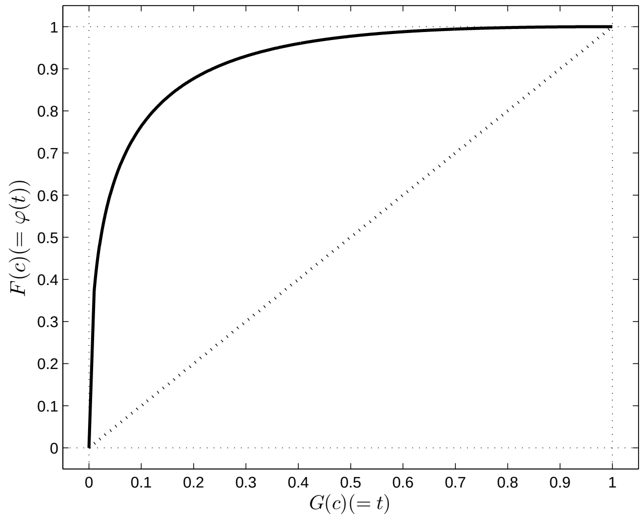

Figure 1: A typical ODC plot.

An obvious non-parametric estimator of $\varphi(t)$ based on data

$$\left\{X_{(1)}<\cdots<X_{(m)},Y_{(1)}<\cdots<Y_{(n)}\right\} \tag{1}$$

is

$$\hat{\varphi}(t)=\hat{F}\left(\hat{G}^{-1}(t)\right) \tag{2}$$

where $\hat{F}$ and $\hat{G}$ denote the empirical distribution functions of the X and Y data:

$$\hat{F}(a)=\frac{1}{m}\sum_{i=1}^{m}I(X_{(i)}\le a)$$

and

$$\hat{G}(a)=\frac{1}{n}\sum_{i=1}^{n}I(Y_{(i)}\le a),$$

$-\infty<a<\infty$, with I denoting the indicator function.

Clearly, both $\varphi(t)$ and $\hat{\varphi}(t)$ remain unchanged if a common strictly increasing function is applied to all the data. This fact has led to the formulation of semi-parametric models in which it is assumed that an unspecified transformation, H, *exists* which makes the distributions of $H(X)$ and $H(Y)$ members of a location-scale family generated by a distribution with known CDF Ψ. That is,

$$\varphi(t) = \Psi\left(\mu + \sigma\Psi^{-1}(t)\right), \tag{3}$$

$-\infty < \mu < \infty$, $\sigma > 0$. It is important to note that there is no assumption here that the X and Y data themselves come from the location-scale family generated by Ψ. The popular bi-normal model (Hanley, 1996) specifies that Ψ is the standard normal CDF Φ. Another model, which may be referred to as the bi–exponential model, has Ψ as a standard exponential cdf on $(0,\infty)$ with $\mu = 0$ and $\sigma > 0$ in (3). In this paper we construct a simultaneous confidence band for the ODC in such a semi-parametric setup and assess its performance by Monte Carlo simulation. Our interest centres on the construction of a *simultaneous* confidence band with specified nominal coverage probability $1-\alpha$. By this is meant that we seek functions $\hat{L}(t)$ and $\hat{U}(t)$ of t and the data (1) with the property that

$$P\left(\hat{L}(t) < \varphi(t) < \hat{U}(t), \text{for all } 0 < t < 1\right) \approx 1-\alpha,$$

or, perhaps less ambitiously,

$$P\left(\hat{L}(t_j) < \varphi(t_j) < \hat{U}(t_j), \text{for } j = 1,\ldots,k\right) \approx 1-\alpha$$

where $0 < t_1 < \cdots < t_k < 1$ and $k > 1$.

Section 2 of the paper describes the methodology underlying construction of our semi–parametric band. Section 3 presents some Monte Carlo simulation results. Section 4 discusses the relation to other non-parametric approaches and identifies two topics for further research. In Section 5 an application to credit scoring data is given and Section 6 summarizes our results.

In order to keep the typography concise, we generally suppress the dependence of quantities on the sample sizes m and n. Thus, we write $\hat{\varphi}(t)$ rather than the more explicit $\hat{\varphi}_{m,n}$. We also mention here that the ODC is a variant of the well known receiver operating characteristic (ROC) curve $1-\varphi((1-t)$. Thus, the confidence bands developed in this paper will, with obvious modifications, also be applicable to ROC estimation. The latter field has recently been thoroughly reviewed by Kranowski and Hand (2009).

2. Construction of a Semi-Parametric Band

We first give an expression for the difference between the empirical *ODC*

$$\hat{\varphi}(t) = \hat{F}\left(\hat{G}^{-1}(t)\right)$$

and its semi-parametric counterpart $\varphi(t)$ from (3). In fact, from Theorem 2.2 of Hsieh and Turnbull (1996), we deduce that for $0 < t_1 < t_2 < 1$,

$$\sup_{t_1 \le t \le t_2} \left|\hat{F}\left(\hat{G}^{-1}(t)\right) - \varphi(t)\right| \approx \frac{1}{\sqrt{m+n}} \sup_{t_1 \le t \le t_2} |\kappa(t)| \tag{4}$$

in distribution for large m and n, where

$$\kappa(t) = \sqrt{\lambda+1}B_1(\varphi(t)) - \varphi'(t)\sqrt{\frac{\lambda+1}{\lambda}}B_2(t)$$

with $0 < \lambda = lim_{m,n\to\infty} n/m < 1$ and B_1 and B_2 are independent Brownian bridges. Let the constant C_α satisfy the relation

$$P\left(\sup_{t_1\le t\le t_2} |\kappa(t)| \le C_\alpha\right) = 1-\alpha. \tag{5}$$

Then, by (4),

$$P\left(\sup_{t_1\le t\le t_2} |\hat{\varphi}(t) - \varphi(t)| \le \frac{C_\alpha}{\sqrt{m+n}}\right) \approx 1-\alpha \tag{6}$$

for large values of m and n. Thus, with

$$\hat{L}(t) = max\left\{0, \hat{\varphi}(t) - \frac{C_\alpha}{\sqrt{m+n}}\right\},\ \hat{U}(t) = min\left\{1, \hat{\varphi}(t) + \frac{C_\alpha}{\sqrt{m+n}}\right\} \tag{7}$$

we potentially have a simultaneous confidence band of very simple form, *provided* we can solve for C_α from (5). Lombard (2005) gives the following result that can be used to this end.

Lemma Let $\kappa(t)$, $0 < t < 1$, be a path-continuous Gaussian process with covariance function

$$C(\varepsilon, t) = Cov(\kappa(t), \kappa(t+\varepsilon)).$$

Suppose there exist continuous functions $\varsigma(t) > 0$ and $\tau(t)$ such that

$$C(\varepsilon, t) = \varsigma(t) - \tau(t)\varepsilon + o(\varepsilon)$$

as $\varepsilon \to 0$. Set

$$h_{C_\alpha}(x) = \begin{cases} \left(2\pi x^3\right)^{-1/2} C_\alpha \exp\left(-C_\alpha^2/2x\right), & x > 0 \\ 0, & x = 0. \end{cases}$$

Then,

$$P\left(\sup_{t_1\le t\le t_2} |\kappa(t)| \ge C_\alpha\right) \sim 2\int_{t_1}^{t_2} |\tau(t)|\, h_{C_\alpha}(\varsigma(t))dt, \tag{8}$$

for small values of α and t_1 and t_2 close to 0 and 1 respectively, where $\sim$ means that the ratio of the two sides tends to 1 as α tends to 0.

The approximation in (8), which does not involve the sample sizes m and n, is quite accurate whenever α is less than 0.1. In our applications the right hand side of (8) is often indeterminate at $t_1 = 0$ and $t_2 = 1$, in which case we take $t_1 = 0.001$ and $t_2 = 0.999$ or $t_1 = 0.01$ and $t_2 = 0.99$. However, the approximation is typically not good for $t_1 = 0.1$ and $t_2 = 0.9$.

A Taylor expansion around $\varepsilon = 0$ shows that the covariance function of the Gaussian process $\kappa(t)$ from (2) has the form

$$\mathbb{C}ov(\kappa(t), \kappa(t+\varepsilon)) = \varsigma(t) - \tau(t)\varepsilon + o(\varepsilon),$$

as $\varepsilon \to 0$, where

$$\varsigma(t) = (\lambda+1)\,\varphi(t)(1-\varphi(t)) + \left(\frac{\lambda+1}{\lambda}\right) t(1-t)\varphi'(t)^2, \tag{9}$$

and

$$\tau(t) = (\lambda+1)\,\varphi(t)\varphi'(t) + \left(\frac{\lambda+1}{\lambda}\right) t\varphi'(t)^2 - \left(\frac{\lambda+1}{\lambda}\right) t(1-t)\varphi'(t)\varphi''(t). \tag{10}$$

Under the semi-parametric specification (3), there are analytic expressions for the derivatives of φ in terms of just two parameters, μ and σ. Denoting by ψ the density function of Ψ, straightforward calculation gives

$$\varphi'(t) = \sigma \frac{\psi\left(\mu + \sigma\Psi^{-1}(t)\right)}{\psi\left(\Psi^{-1}(t)\right)} \tag{11}$$

and

$$\varphi''(t) = \frac{\varphi'(t)}{\psi\left(\Psi^{-1}(t)\right)} \left(\sigma \frac{\psi'\left(\mu + \sigma\Psi^{-1}(t)\right)}{\psi\left(\mu + \sigma\Psi^{-1}(t)\right)} - \frac{\psi'\left(\Psi^{-1}(t)\right)}{\psi\left(\Psi^{-1}(t)\right)} \right). \tag{12}$$

The parameters μ and σ can be estimated $\sqrt{m+n}$-consistently by, for instance, a version of the non-parametric minimum distance method developed by Hsieh and Turnbull (1996). In the present context, this entails minimizing the expression

$$Q(\mu,\sigma) = \sum_{j=1}^{n} \left(\hat{F}\left(\hat{G}^{-1}\left(\frac{j}{n+1}\right)\right) - \Psi\left(\mu + \sigma\Psi^{-1}\left(\frac{j}{n+1}\right)\right)\right)^2$$

over μ and σ.

Denote by $\hat{\varsigma}(t)$ and $\hat{\tau}(t)$ the quantities $\varsigma(t)$ and $\tau(t)$ in (9) and (10) after substitution of the estimates of μ and σ into the expressions in (11) and (12). C_α is then estimated via the right hand side of (8) by the random variable $\hat{C}_\alpha$, defined as the solution in x to the equation $\gamma(x) = \alpha/2$, where

$$\gamma(x) = \int_{t_1}^{t_2} |\hat{\tau}(t)|\, h_x(\hat{\varsigma}(t))\, dt. \tag{13}$$

Replacing C_α in (7) by $\hat{C}_\alpha$ results in a confidence band with lower and upper limits of a very simple form,

$$\hat{L}(t) = max\left\{0, \hat{F}(\hat{G}^{-1}(t)) - \frac{\hat{C}_\alpha}{\sqrt{m+n}}\right\},\ \hat{U}(t) = min\left\{1, \hat{F}(\hat{G}^{-1}(t)) + \frac{\hat{C}_\alpha}{\sqrt{m+n}}\right\},$$

$0 < t_1 < t_2 < 1$.

In practice, equation (13) will be solved numerically. Our experience with the simulation results reported in Section 3 indicates that it is typically sufficient to take $t_1 = 1 - t_2 = 0.001$. (One cannot take $t_1 = 0$, $t_2 = 1$ because the integrand is infinite there.) For an easily computable form of $\hat{F}\hat{G}^{-1}(t)$, denote by R_i the rank of $X_{(i)}$ in the data set (1). Then it is readily shown that

$$\hat{F}(\hat{G}^{-1}(t)) = \frac{i-1}{m} \text{ for } \frac{R_{i-1}-(i-1)}{n} < t < \frac{R_i - i}{n} \text{ and } 1 \le i \le m+1,$$

with $R_0 = 0$ and $R_{m+1} = \infty$.

3. Monte Carlo Simulation Results

We now assess the performance of the semi-parametric band by estimating its true coverage probability via Monte Carlo simulation. Under the model (3), the minimum distance estimators of μ and σ are invariant under any strictly monotonic transformation of the data. In the simulations it is therefore sufficient to restrict attention to data from a location-scale family $\Psi((x-\mu)/\sigma)$, provided that μ and σ are estimated by a non-parametric method such as the minimum distance method (and not by maximum likelihood using the given form of Ψ). As a measure of separation between the X and Y distributions we use the "area under the curve"

$$A = \int_0^1 \varphi(t)dt = P(X < Y) \tag{14}$$

which is related to the well known Gini coefficient of inequality. We present here an extract of results from a Monte Carlo study which involved the bi-normal and bi-exponential models.

First, data were generated from normal $(0,1)$ and normal (μ, σ) distributions where, for given σ, μ is chosen to give the area under the curve, A, in (14) a specified value:

$$\mu = \sqrt{1+\sigma^2}\Phi^{-1}(A).$$

$N = 2,000$ Monte Carlo trials were run to estimate the true coverage probabilities. The results are in Table 1, in which β denotes the nominal coverage probability. The results are reported for three values of σ and three AUC values, the latter three being typical of what we have seen in practice.

Table 1: Estimated coverage probabilities in some bi-normal models.

		$\beta = 0.90$			$\beta = 0.95$		
		AUC			AUC		
	$m = n = 500$	0.6	0.7	0.8	0.6	0.7	0.8
σ	1.5	0.91	0.91	0.92	0.96	0.95	0.96
	2.0	0.90	0.89	0.91	0.95	0.95	0.96
	2.5	0.89	0.88	0.91	0.96	0.95	0.96
		$\beta = 0.90$			$\beta = 0.95$		
		AUC			AUC		
	$m = n = 250$	0.6	0.7	0.8	0.6	0.7	0.8
σ	1.5	0.89	0.91	0.91	0.95	0.95	0.96
	2.0	0.89	0.90	0.92	0.95	0.96	0.96
	2.5	0.89	0.91	0.91	0.95	0.94	0.96
		$\beta = 0.90$			$\beta = 0.95$		
		AUC			AUC		
	$m = n = 100$	0.6	0.7	0.8	0.6	0.7	0.8
σ	1.5	0.91	0.93	0.94	0.95	0.97	0.97
	2.0	0.92	0.90	0.93	0.96	0.96	0.96
	2.5	0.91	0.92	0.92	0.97	0.95	0.97

Next, data were generated from the scale parameter family of exponential distributions with distribution functions $1 - exp(-x/\sigma)$, $x, \sigma > 0$ in which the ODC is $\varphi(t) = 1 - (1-t)^\sigma$. Here, the choice $\sigma = A/(1-A)$ produces an AUC equal to A. The results are in Table 2.

Table 2: Estimated coverage probabilities in the bi-exponential model.

	$\beta = 0.90$			$\beta = 0.95$		
	AUC			AUC		
	0.6	0.7	0.8	0.6	0.7	0.8
$m = n = 500$	0.91	0.90	0.91	0.95	0.96	0.97
$m = n = 250$	0.91	0.91	0.91	0.96	0.96	0.96
$m = n = 100$	0.92	0.93	0.93	0.96	0.96	0.96

These results, which are representative of a general picture, indicate that the (Monte Carlo estimated) true coverage probabilities are generally sufficiently close to the nominal values for practical use when both sample sizes are large. In almost all such instances the estimated and nominal coverage probabilities differ by less than 0.01. The true coverage probabilities consistently exceed the nominal values (by 0.01) at the higher value of A. In fact, the difference becomes even larger when $A = 0.9$ or 0.95, which is indicative of near perfect separation of the distributions. This is easy to understand if account is taken of the form of the ODC in such cases. With $A = 0.8$ and $\sigma = 1.5$ the bi-normal ODC is effectively equal to its maximum value of 1 for all $t \geq 0.7$. The true coverage probability over the range $t \geq 0.7$ is therefore effectively $1 - \alpha/2$, which is larger than the nominal $1 - \alpha$. The effect is greater at the smaller sample sizes $m = n = 100$ because the band is then so wide that the upper (lower) limit at $A \geq 0.7$ becomes effectively equal to 1 (0) for $t \geq .0.6$ ($t \leq 0.1$). However, in the credit risk applications that we have encountered, AUCs in excess of 0.8 are extremely rare, as are samples of size less than 250.

4. Discussion

In a fully distribution-free setup, Horváth, Horváth and Zhou (2008) use a version of (6) and estimate C_α by the application of a smooth bootstrap. Horváth et al. (2008) evaluate their bootstrap method by Monte Carlo simulation on data generated from an exponential distribution and report, without providing numerical details, that it performs satisfactorily in that particular instance. Li, Tiwari and Wells (1996) also provide a bootstrap approach to the more complex problem when censored data are involved and find it to be satisfactory when the data come from a Weibull distribution. However, in the context of finding a confidence interval at a single point, t, Hall, Hyndman and Fan (2004) find that the success of the bootstrap is critically dependent on a correct choice of certain smoothing parameters. If the form of the underlying distributions is known, then presumably the correct choices can be made. It is not clear, however, how successful a fully non-parametric bootstrap would be.

The level of technical expertise required for such bootstrapping in each individual instance limits somewhat the possibility of automated and routine application in a typical credit scoring environment. For a possible alternative approach, we note that the right hand sides in (9) and (10) involve derivatives of the non-parametric ODC $\varphi(t) = F(G^{-1}(t))$, namely

$$\varphi'(t) = \frac{f(G^{-1}(t))}{g(G^{-1}(t))}$$

and

$$\varphi''(t) = \frac{f'(G^{-1}(t)) - g'(G^{-1}(t))\varphi'(t)}{g^2(G^{-1}(t))}.$$

In a fully distribution-free setup, and if sufficiently large samples were available, the densities and their derivatives could be estimated by a fully automated procedure and substituted into the right hand sides of (9) and (10). This would lead to a fully non-parametric band that does not require bootstrapping or an assumption about the parametric form of the ODC. This possibility certainly merits further investigation.

The approach we have followed is of a hybrid nature in that a fully non-parametric estimator is being compared with an estimated semi-parametric model. The advantage in this lies in the fact that the latter model has dimension equal to 2 — two parameters, both of which can be efficiently estimated from a limited amount of data — while a fully nonparametric model is infinite-dimensional. A fully semi-parametric approach could presumably be based on a probability statement such as

$$P\left(sup_{a<t<b}\left|\Psi\left(\hat{\mu}+\hat{\sigma}\Psi^{-1}(t)\right)-\Psi(\mu+\sigma\Psi^{-1}(t))\right|\le c_\alpha\right)\approx 1-\alpha.$$

However, it is not immediately clear how c_α could be determined since the stochastic process $\Psi\left(\hat{\mu}+\hat{\sigma}\Psi^{-1}(t)\right)$ does not seem to have a tractable asymptotic limiting form. As a first approximation one could take

$$\Psi\left(\hat{\mu}+\hat{\sigma}\Psi^{-1}(t)\right)-\Psi(\mu+\sigma\Psi^{-1}(t))\approx\frac{(\hat{\mu}-\mu)+(\hat{\sigma}-\sigma)\Psi^{-1}(t)}{\psi\left(\Psi(\mu+\sigma\Psi^{-1}(t))\right)}.$$

Then, the expression on the right hand side, multiplied by $\sqrt{m+n}$, converges to a Gaussian process with a rather complicated covariance function — see Hsieh and Turnbull (1996, Theorem 3.2).

The question of whether a semi-parametric model such as (3) can reasonably be assumed to hold in any specific application seems not to have received much attention in the literature. However, Hanley (1996) concludes that the use of the bi-normal model is widely justified. A simple diagnostic consists in seeing whether the estimated semi-parametric curve is wholly contained in the confidence band we have constructed.

5. Application to Credit Scoring Data

Credit risk is the potential financial loss that lenders may incur owing to unexpected changes in the credit quality of obligors. Credit risk assessment has gained increased popularity since the advent of the global financial crisis in 2008/2009. Lending institutions are particularly concerned with understanding the risk profile of their existing and potential client base. Risk assessment superiority directly translates into competitive advantage. Once lenders are able to understand the risks that they are faced with, informed decisions can be made with regards to price setting, loan specifications and loan agreements. Credit providers typically use scorecards (sometimes referred to as rating systems) to assess the credit worthiness of a client. Characteristics of the client believed to be indicative of risk are summarized by the scorecard. The scorecard translates the client information into a score. Typically, scorecards are constructed such that low risk clients receive a high score and high risk clients receive a low score. To judge the efficacy of a scorecard, one approach is to relate the credit score of an individual who has been granted credit to the occurrence of default on the obligation.

Represent the credit scores of defaulters and non-defaulters generically by X and Y with distribution functions F and G respectively. The data, available upon request from the second author, consist of credit scores of $m+n=750$ obligors, $m=370$ of whom were defaulters. Figure 2 is a plot of $\gamma(x)$ from (13) against x for the bi-normal model using $t_1=0.001$ and $t_2=0.999$. This gives $\hat{C}_{0.05}=3.23$. The non-parametric ODC estimate (2), the estimated bi-normal curve, $\Phi\left(0.94+0.96\times\Phi^{-1}(t)\right)$ and the 95% confidence band are shown in Figure 3. Since the estimated bi-normal curve is wholly contained in the band, the indication is that the bi-normal assumption is reasonable.

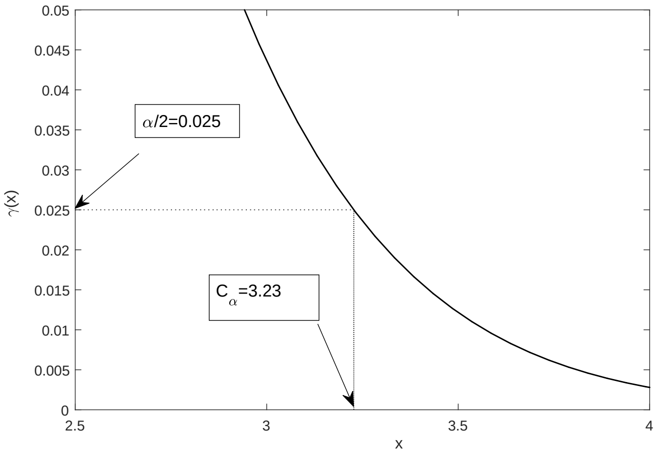

Figure 2: Illustrating the solution to the equation $\gamma(x)=\alpha/2$ for $\gamma(x)$ defined in equation (17) and $\alpha=0.05$.

Figure 4, on the other hand, shows these plots for the bi-exponential model

$$\varphi(t)=1-(1-t)^{\mu}$$

which comes from the choice $\Psi(x)=1-exp(-\mu x)$ in (3) and for which we find $\hat{\mu}=3.16$ and $\hat{C}_{0.05}=1.38$. Notice that the estimated bi-exponential ODC lies outside the band at both lower and higher values of t, which suggests strongly that the model is inappropriate. Since all the data are positive, one may have expected the model to be appropriate, but this is not the case.

6. Summary

We develop a simultaneous confidence band for an ordinal dominance curve under a semi-parametric specification. Such a curve is useful in assessing the efficacy of a diagnostic test that seeks to correctly classify subjects into distinct groups. For instance, credit scorecards are often used to classify applicants for credit into high or low risk categories. The band developed in this paper is easy to implement and exhibits satisfactory performance. Application of the method is illustrated on data from a credit rating scorecard.

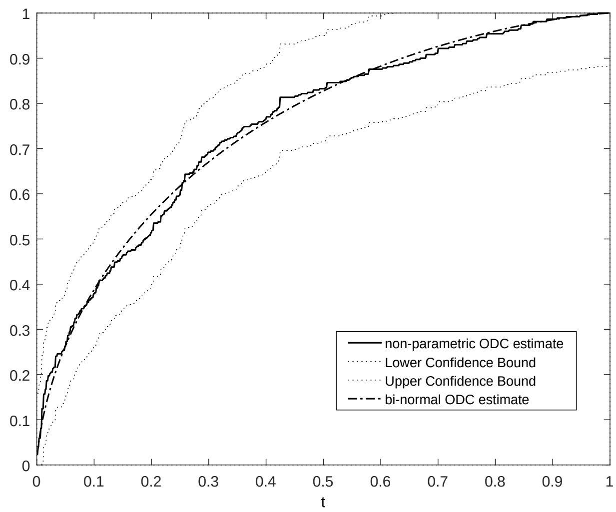

Figure 3: Non-parametric and estimated bi-normal ODC together with 95% simultaneous confidence band.

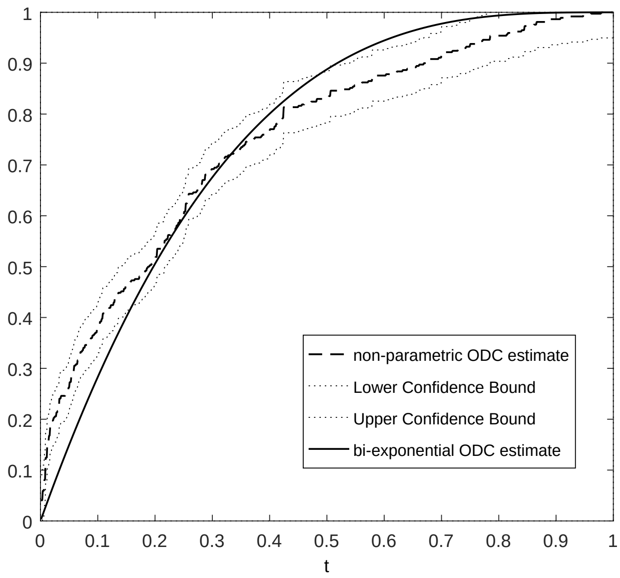

Figure 4: Non-parametric and estimated bi-exponential ODC together with 95% simultaneous confidence band.

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